Unique definition of relative velocity of luminous source as measured along the observer's line-of-sight in a pseudo-Riemannian space-time

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Abstract

Using a way of separating the spectral shifts into infinitesimally displaced `relative´ spectral bins and sum over them, we overcome the ambiguity of the parallel transport of four-velocity, in order to give an unique definition of the so-called *kinetic* relative velocity of luminous source as measured along the observer's line-of-sight in a generic pseudo-Riemannian space-time. The ubiquitous relationship between the spectral shift and the *kinetic* relative velocity is utterly distinct from a familiar global Doppler shift rule (Synge, 1960). Such a performance of having found a *kinetic* relative velocity of luminous source, without subjecting it to a parallel transport, manifests its virtue in particular case when adjacent observers are being in free fall and populated along the null geodesic, so that it is reduced to a global Doppler velocity as studied by Synge. We discuss the implications for the instructive case of spatially homogeneous and isotropic Robertson-Walker space-time, which leads to cosmological consequences that the resulting *kinetic* recession velocity of a galaxy is always subluminal even for large redshifts of order one or more, and thus, it does not violate the fundamental physical principle of *causality*.

Keywords: *Classical general relativity; Riemannian space-time; Relative velocity*

1. Introduction

For test particles and observers there is no unique way to compare four-vectors of the velocities at widely separated space-time events in a curved Riemannian space-time, because general relativity (GR) provides no *a priori* definition of relative velocity. This inability to compare vectors at different points was the fundamental feature of a curved space-time. Different coordinate reference frames and notions of relative velocity yield different results for the motion of distant test particles relative to a particular observer. The three distinct coordinate charts are employed by Bolós (2006, 2007), Bolós & Klein (2012), Bolós et al. (2002), Klein & Collas (2010), Klein & Randles (2011), each with different notions of simultaneity, to calculate the four geometrically defined inequivalent concepts of relative velocity: Fermi, kinematic, astrometric, and the spectroscopic relative velocities. The four definitions of relative velocities depend on two different notions of simultaneity: "spacelike simultaneity" (or "Fermi simultaneity") (Klein & Randles, 2011, Walker, 1935) as defined by Fermi coordinates of the observer, and "lightlike simultaneity" as defined by optical (or observational) coordinates of the observer (Ellis, 1985). The Fermi and kinematic relative velocities can be described in terms of the "Fermi simultaneity", according to which events are simultaneous if they lie on the same space slice determined by Fermi coordinates. Thereby, for an observer following a timelike worldline in Riemannian space-time, Fermi-Walker coordinates provide a system of locally inertial coordinates. If the worldline is geodesic, the coordinates are commonly referred to as Fermi or Fermi normal coordinates. Useful feature of Fermi coordinates was that the metric tensor expressed in these coordinates is Minkowskian to first order near the path of the Fermi observer, with second order corrections involving only the curvature tensor (Manasse & Misner, 1963). In (Klein & Randles, 2011), the authors find explicit expressions

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for the Fermi coordinates for Robertson-Walker (RW) space-times and show that the Fermi chart for the Fermi observer in non-inflationary RW space-times is global. However, rigorous results for the radius of a tubular neighborhood of a timelike path for the domain of Fermi coordinates are not available. The spectroscopic (or barycentric) and astrometric relative velocities, which can be derived from spectroscopic and astronomical observations, mathematically, both rely on the notion of light cone simultaneity. According to the latter, two events are simultaneous if they both lie on the same past light cone of the central observer. It is shown that the astrometric relative velocity of a radially receding test particle cannot be superluminal in any expanding RW space-time. Necessary and sufficient conditions are given for the existence of superluminal Fermi speeds. Note that for the Hubble velocity, the proper distance is measured along non geodesic paths, while for the Fermi velocity, the proper distance is measured along spacelike geodesics. In this respect the Fermi velocity seems to be more natural, but the Hubble velocity is defined at all space-time points, whereas the Fermi velocity makes sense only on the Fermi chart of the central observer. Although alluded four definitions of relative velocities have own physical justifications, all they are subject to many uncertainties, and the ambiguity still remains.

Keeping in mind aforesaid, below we restrict our analysis to seeking solution for particular case when a test particle is being a luminous object. In this case, the problem of a definition of relative velocity can be significantly simplified because of available spectral shift measured by observer. The hope appears that a relative velocity of luminous source as measured along the observer's line-of-sight (speed) can be defined in unique way straightforwardly from kinematic spectral shift rule, which holds on a generic pseudo-Riemannian manifold (Synge, 1960). In the same time, aforementioned inability to immediately compare the four-velocity vector V_S^μ of the luminous object S with the four-velocity vector V_O^μ of the observer O in pseudo-Riemannian space-time necessitates to seek a useful definition of the relative velocity by bringing both vectors to a common event. Historically, Synge has subjected the vector V_S^μ to parallel transport along the null geodesic to the observer, and deduced a global Doppler effect in terms of energy and frequency. It is well known that null geodesics are peculiar, in a sense that they lie in a metric space wherein they are being only 1D-affine spaces, so that only a parallel displacement, not a metric distance is defined along them. As a corollary, their geometric properties become a rather unexpected mixture of affine and metric properties. At first glance, we seem to have attractive proposal of choosing a null geodesics for the parallel transport since it does not require any additional structures, like particular foliation of space-time, which in turn is applicable to any space-time. However, a resulting Doppler effect is inconclusive, because a definition of relative velocity has disadvantage that there is no unique way to compare four-vectors of the velocities at widely separated space-time events by parallel transport.

Our primary interest, in the present article is rather to extend those geometrical ideas developed by (Synge, 1960), to build a series of infinitesimally displaced shifts and then sum over them in order to achieve an unique definition of the so-called *kinetic* relative velocity of luminous source, without subjecting it to a parallel transport, as measured along the observer's line-of-sight in a generic pseudo-Riemannian space-time. These peculiarities deserve careful study, because they furnish valuable theoretical clues about the interpretation of kinetic velocity of luminous object relative to observer in GR, a systematic analysis of properties of which happens to be surprisingly difficult by conventional methods. The problem of subjecting the four-velocity vector of the luminous source to parallel transport will not be broached in this paper, though it is hoped that the present formulation of the theory will facilitate the task. A resulting general relationship between the spectral shift and the *kinetic* relative velocity is utterly distinct from a familiar global Doppler shift. We discuss the implications for a particular case of a global Doppler shift along the null geodesic, and the spatially homogeneous and isotropic Robertson-Walker space-time of standard cosmological model. The latter leads to important cosmological consequences that the resulting *kinetic* recession velocity of a galaxy is always subluminal even for large redshifts of order one or more, and thus, it does not violate the fundamental physical principle of *causality*. This provides a new perspective to solve startling difficulties of superluminal `proper´ recession velocities, which the conventional scenario of expanding universe of standard cosmological model presents (see e.g. Bolós & Klein, 2012, Bunn & Hogg, 2009, Chodor-

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owski, 2011, Davis & Lineweaver, 2004, Emtsova & Toporensky, 2019, Grøn & Elgarøy, 2007, Harrison, 1993, 1995, Kaya, 2011, Klein & Collas, 2010, Klein & Randles, 2011, Murdoch, 1977, Narlikar, 1994, Peacock, 1999, 2008, Peebles, 1993, Peebles et al., 1991, Silverman, 1986, Whiting, 2004). In some instances (in earlier epochs), the distant astronomical objects are observed to exhibit redshifts in excess of unity, and only a consistent theory could tackle the key problems of a dynamics of such objects.

With this perspective in sight, we will proceed according to the following structure. To start with, Section 2 deals with the relationship of the overall spectral shift, z, and the speed of source (S) relative to observer (O) in its rest frame along the line of sight, in a general Riemannian space-time. On these premises, in Subsection 2.1, we show that a general solution is reduced to global Doppler shift along the null geodesic, as it is studied by Synge (1960). We use a general solution as a backdrop to explore in Subsection 2.2 a new concept of the *kinetic* recession velocity of an astronomical object in other context of RW space-time of standard cosmological model. In these terms we reconcile the cosmological interpretation of redshift with the correct solution to a kinematic interpretation of redshifts as accumulation of a series of infinitesimal spectral shifts. The *kinetic* recession velocity is always subluminal even for large redshifts of order one or more, and thus, it does not violate the fundamental physical principle of *causality*. Concluding remarks are given in Section 3.

2. The relative speed along the observer's line-of-sight in a generic pseudo-Riemannian space-time

Avoiding from any mistakes, we preferred to work in an infinitesimal domain. The principle foundation of our setup comprises the following steps. Let (o) and (s) be two world lines respectively of observer O and source S in the pseudo-Riemannian space-time. Suppose the passage of light signals from S to O is described by a single infinity of null geodesics $\Gamma(v)$ connecting their respective world lines. To clarify the issues further, it should help a few noteworthy points of Fig. 1. Avoiding from any mistakes, we preferred to work in an infinitesimal domain. The $S_{(1)}$ and $S_{(2)}$ are two neighboring world points on (s). The parametric values for these geodesics are $v, v+\vartriangle v$, respectively, where $v = const$ and $\vartriangle v$ is infinitesimally small. Accordingly, the world line (s) is mapped pointwise on the (o) by a set of null geodesics $\Gamma(v)$. That is, a set of null geodesics are joining (s) to (o), each representing the history of a wave crest. The totality of these null geodesics forms a 2-space with equation $x^\mu = x^\mu(u, v)$, which is determined once (s) and (o) are given. The u denote the affine parameter on each of these geodesics running between fixed end-values $u = 0$ on (s) and $u = 1$ on (o). The $O_{(1)}$ and $O_{(2)}$ are corresponding world points on (o), where the null geodesics from $S_{(1)}$ and $S_{(2)}$ meet it. Also we will denote by τ_O and τ_S the proper times of the observer and the source, respectively, and $\vartriangle\tau_O$ and $\vartriangle\tau_S$ are the elements of proper time corresponding to the segments (the clock measures of) $O_{(1)}O_{(2)}$ and $S_{(1)}S_{(2)}$. Imagine now a dense family of adjacent observers O_j $(j = 1, ..., n-1)$ with the world lines (o_j) populated between the two world lines (o) and (s). Each observer O_j measures the frequency of light rays emitted by the source S as it goes by. The $O_{j(1)}$ and $O_{j(2)}$ are two neighboring world points on (o_j) where the null geodesics from $S_{(1)}$ and $S_{(2)}$ meet it. The u_j denote the values of affine parameter on each of the null geodesics chosen at equal infinitesimally small δu_i, so that $u = u_j$ on (o_j). The τ_{O_j} denotes the proper times of the adjacent observers, i.e. $\vartriangle\tau_{O_j}$ are the elements of proper time corresponding to the segment $O_{j(1)}O_{j(2)}$. Here and throughout we use the proper space scale factor l_i $(i = 0, 1, 2, ..., n)$ which encapsulates the beginning and evolution of the elements of proper time $\vartriangle\tau_S$ of source, namely $l_0 = c\vartriangle\tau_S$, $l_1 = c\vartriangle\tau_{O_{(1)}}$, ... , $l_{n-1} = c\vartriangle\tau_{O_{n-1}}$, $l_n = c\vartriangle\tau_O$. Each line segment l_{i-1} of proper space scale factor (at the affine parameter u_{i-1}) is identically mapped on the line segment $(l_i - \delta l_{i-1})$ of proper space scale factor (at infinitesimally close affine parameter u_i), such that $l_{i-1} \equiv (l_i - \delta l_{i-1})$, where δl_i denotes infinitesimal segment $a_i O_{i(2)}$. If there are N wave crests of the light, the wavelength of light λ_{O_i} at the observers O_i $(i = 1, ..., n$, where $O_n \equiv O)$, who measures the wavelength of light ray as it goes by, satisfies the following condition:

$$N = \frac{l_n}{\lambda_n} = \frac{l_{n-1}}{\lambda_{n-1}} = \cdots = \frac{l_1}{\lambda_1} = \frac{l_0}{\lambda_0}, \tag{1}$$

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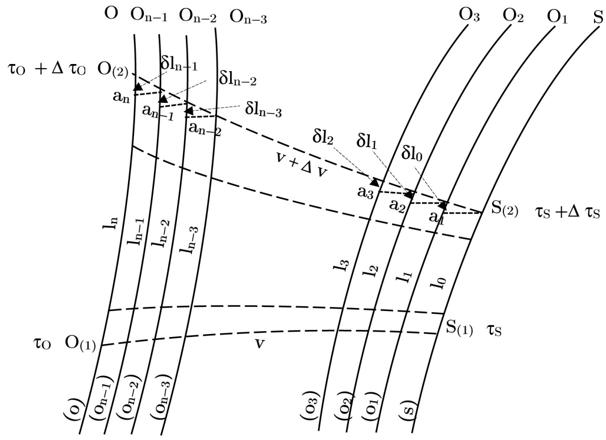

Figure 1. The infinitesimal spectral shifts as measured locally by emitter and adjacent receivers in a generic pseudo-Riemannian space-time. The (o) and (s) are two world lines respectively of observer O and source S. A dense family of adjacent observers O_j $(j = 1, ..., n-1)$ with the world lines (o_j) populated between the two world lines (o) and (s). A set of null geodesics (the dotted lines) is mapping (s) on (o), each representing the history of a wave crest. Each line segment l_{i-1} of proper space scale factor (at the affine parameter u_{i-1}) is identically mapped on the line segment $(l_i - \delta l_{i-1})$ of proper space scale factor (at infinitesimally close affine parameter u_i), such that $l_{i-1} \equiv (l_i - \delta l_{i-1})$, where δl_i denotes infinitesimal segment $a_i O_{i(2)}$.

where $\lambda_i \,(\equiv \lambda_{O_i})$. The wavelength λ_i of light ray is varied on the infinitesimal distance between the observers O_{i+1} and O_i in a general Riemannian space-time in proportion to the proper space scale factor l_i :

$$\frac{\lambda_{i+1}}{\lambda_i} = \frac{\triangle\tau_{O_{i+1}}}{\triangle\tau_{O_i}} = \frac{l_{i+1}}{l_i}, \tag{2}$$

such that the spectral shift z_i reads

$$z_i = \frac{\lambda_i}{\lambda_S} - 1 = \frac{l_i}{l_0} - 1 = \frac{\triangle\tau_{O_i}}{\triangle\tau_S} - 1. \tag{3}$$

The spectral shift z_i, in general, can be evaluated straightforwardly in terms of the world function $\Omega(SO_i)$ for two points $S(x')$ and $O_i(x_{(i)})$ $(i = 1, ..., n)$ through an integral defined along the geodesic $\Gamma_{SO_i}(v)$ joining them (Synge, 1960), taken along any one of the curves $v = const$. Following Synge, the world function $\Omega(SO_i)$ can be defined for any of the geodesics in the family linking points on (o_i) and (s):

$$\Omega(SO_i) = \Omega(x' x_{(i)}) \equiv \Omega_i(v) = \tfrac{1}{2}(u_{Oi} - u_S) \int_{u_S}^{u_{Oi}} g_{\mu\nu} U^\mu U^\nu du, \tag{4}$$

taken along $\Gamma_{SO_i}(v)$ with $U^\mu = \frac{dx^\mu_{(i)}}{du}$, has a value independent of the particular affine parameter chosen. The holonomic metric $g = g_{\mu\nu}\,\vartheta\mu \otimes \vartheta^\nu = g(e_\mu, e_\nu)\,\vartheta^\mu \otimes \vartheta^\nu$, is defined in the Riemannian space-time, with the components, $g_{\mu\nu} = g(e_\mu, e_\nu)$ $(\mu = 0, 1, 2, 3)$ in the dual holonomic base $\{\vartheta^\mu \equiv dx^\mu\}$. We have taken $u_S = 0$ and $u_{Oi} \leq 1$ for given world function $\Omega_{(i)}(v)$, which becomes

$$\Omega_{(i)}(v) = \tfrac{1}{2} u_{Oi} \int_0^1 g_{\mu\nu} U^\mu U^\nu du. \tag{5}$$

By virtue of $\delta U^\mu/\delta u = 0$, we have $g_{\mu\nu} U^\mu U^\nu = const$ along $\Gamma_{SO_i}(v)$, therefore, (4) is reduced to

$$\Omega_{(i)}(v) = \tfrac{1}{2}(u_{O_{(i)}} - u_S)^2 g_{\mu\nu} U^\mu U^\nu, \tag{6}$$

with the last part evaluated anywhere on $\Gamma_{SO_i}(v)$. Taking $u_S = 0$ and $u_{Oi} = 1$, and applying conventional methods (Synge, 1960), we then have

$$\Omega_{(i)}(v) = \tfrac{1}{2} g_{\mu\nu} U^\mu U^\nu = \tfrac{1}{2}\varepsilon L_i^2, \quad L_i = \int_S^{O_i} ds, \tag{7}$$

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that is, to within the factor $\varepsilon=\pm 1$, the world-function is half the square of the measure, L_i, of geodesic joining S and O_i. Synge has proved that the following relations hold in general:

$$\frac{\partial\Omega_{(i)}(v)}{\partial x^\mu}\Big|_S = -u_{Oi}\, g_{\mu\nu}\frac{\partial x^\nu}{\partial u}\big|_S\,, \quad \frac{\partial\Omega_{(i)}(v)}{\partial x^\mu}\Big|_{O_i} = u_{Oi}\, g_{\mu\nu}\frac{\partial x^\nu}{\partial u}\big|_{O_i}\,. \tag{8}$$

The right hand sides are invariant under transformation of the affine parameter. If the geodesic is not null, one has

$$\frac{\partial\Omega_{(i)}}{\partial x'^\mu} = -L_i\tau_{\mu(S)}, \quad \frac{\partial\Omega_{(i)}}{\partial x^\mu_{(i)}} = L_i\tau_{\mu(O_i)}, \tag{9}$$

where $\tau_{\mu(S)}$ and $\tau_{\mu(O_i)}$ are the unit tangent vectors to the geodesic at S and O_i.

For null geodesics $\Gamma_{S_{(1)}O_{i(1)}}(v)$ and $\Gamma_{S_{(2)}O_{i(2)}}(v+\triangle v)$, in particular, the world functions $\Omega_{(i)}(v)$ does not change in the interval v and $v+\triangle v$, therefore

$$\frac{\partial\Omega_{(i)}(v)}{\partial x^\mu}\frac{dx^\mu}{dv}\Big|_{O_i} + \frac{\partial\Omega_{(i)}(v)}{\partial x^\mu}\frac{dx^\mu}{dv}\Big|_S = 0, \tag{10}$$

which yields

$$p_{\mu(i)}V^\mu_{(i)}\triangle\tau_{O_i} - p_{\mu(S)}V^\mu_{(S)}\triangle\tau_S = 0, \tag{11}$$

where $V^\mu_{(i)} = dx^\mu/d\tau_{O_i}|_{O_{i(1)}}$ and $V^\mu_{(S)} = dx^\mu/d\tau_S|_{S_{(1)}}$ are the respective four-velocity vectors of observer O_i and source S (or world lines (o_i) and (s)) at points $O_{i(1)}$ and $S_{(1)}$, $p^\mu_{(i)} = dx^\mu_{(i)}/du_i$ and $p^\mu_{(S)} = dx'^\mu/du_0$ are respective four-momenta of light ray (tangent to null geodesic) at the end points. Then, by virtue of (2), we obtain

$$1+z_i = \frac{l_i}{l_0} = \frac{p_{\mu(S)}V^\mu_{(S)}}{p_{\mu(i)}V^\mu_{(i)}}. \tag{12}$$

For $i=n$, (12) becomes a well-known generalization of the overall spectral shift rule in a Riemannian space-time (Synge, 1960)

$$z = \frac{\triangle\tau_O}{\triangle\tau_S} - 1 = \frac{p_{\mu(S)}V^\mu_{(S)}}{p_{\mu(O)}V^\mu_{(O)}} - 1. \tag{13}$$

Let us subject the vector $V^\mu_{(S)}$ to parallel transport along the null geodesic $\Gamma_{S_{(1)}O_{(1)}}(v)$ to the observer. This yields at $O_{(1)}$ the vector $\beta^\mu_{(S1)} = \mathrm{g}^{\mu\nu'}(O_{(1)},S_{(1)})V^{\nu'}_{(S1)}$, where the two point tensor $\mathrm{g}^{\mu\nu'}(O_{(1)},S_{(1)})$ is the parallel propagator. The latter is determined only by the points $S_{(1)}$ and $O_{(1)}$. At $S_{(1)}\to O_{(1)}$, we have the coincidence limit $[\mathrm{g}^{\mu\nu'}](O_{(1)}) = g^{\mu\nu}(O_{(1)})$. Then we obtain a relativistically invariant form of global Doppler shift:

$$z = \frac{p_{\mu(O1)}\beta^\mu_{(S1)}}{p_{\mu(O1)}V^\mu_{(O1)}} - 1, \tag{14}$$

where $V^\mu_{(O1)}$ is the four-velocity vector of the observer O at point $O_{(1)}$, $p_{\mu(S1)}$ and $p_{\mu(O1)}$ are the tangent vectors to the typical null geodesics $\Gamma_{S_{(1)}O_{(1)}}(v)$ at their respective end points. Narlikar (Narlikar, 1994) has proved a rule (14) in other context of standard cosmological model of expanding universe. A Doppler effect (14), having recast in an alternative form by Synge (1960), reads:

$$z = 1 - \frac{1}{(1+\beta^2_{(O_{(1)})})^{\frac{1}{2}} + \beta_{R(O_{(1)})}}, \tag{15}$$

where $c\beta^\mu_{(S1)} = v^\mu_{(S1)}$, $c\beta_{(O_{(1)})} = v_{(O_{(1)})}$, $c\beta_{R(O_{(1)})} = v_{R(O_{(1)})}$, and

$$\begin{aligned} &v^2_{(O_{(1)})} = v_{(\alpha)(O_{(1)})}v^{(\alpha)}_{(O_{(1)})}, \quad v_{(\alpha)(O_{(1)})} = v_{\mu(S1)}\xi^\mu_{(\alpha)(O_{(1)})}, \\ &v_{R(O_{(1)})} = v_{\mu(S1))}r^\mu_{(O_{(1)})} = v_{(\alpha)(O_{(1)})}v^{(\alpha)}_{(O_{(1)})}. \end{aligned} \tag{16}$$

Reviewing notations the three-velocity of (s) relative (o) are defined by the tree invariant components $v_{(\alpha)(O_{(1)})}$, $v_{(S1)}$ is the relative speed, $v_{R(O_{(1)})}$ is the speed of recession of (s). Whereas $\xi^\mu_{(\alpha)(O_{(1)})}$ is the frame of reference on world-line (o) with $\xi^\mu_{0(O_{(1)})} = V^\mu_{(O_{(1)})}$, the unit vector $r^{\mu(O_{(1)})}$ at $O_{(1)}$ is orthogonal

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to world-line (o) $(r_{\mu(O_{(1)})}V^{\mu}_{(O_{(1)})}=0)$ and lying in the 2-element which contains the tangent at $O_{(1)}$ to (o) and $S_{(1)}O_{(1)}$.

In studying further a set of null geodesics $\Gamma(v)$ with equations $x^{\mu}(u_i,v)$ (where $v=const$), we may deal with the deviation vector $\eta^{\mu}_{(i)}$ drawn from $O_{i(1)}S_{(1)}$ to $O_{i(2)}S_{(2)}$, and that we have along null geodesic

$$\eta_{\mu(i)}\frac{\partial x^{\mu}}{\partial u_i}=const. \tag{17}$$

The equation (17) yields

$$\eta_{\mu(i+1)}p^{\mu}_{(i+1)}=\eta_{\mu(i)}p^{\mu}_{(i)}. \tag{18}$$

Then

$$\begin{aligned}\eta_{\mu(i+1)}&=V^{\mu}_{(i+1)}l_{i+1},\quad \eta_{\mu(i)}=V^{\mu}_{(i)}l_i,\\ \eta_{\mu(i+1)}p^{\mu}_{(i+1)}&=E_{i+1}l_{i+1},\quad \eta_{\mu(i)}p^{\mu}_{(i)}=E_il_i,\end{aligned} \tag{19}$$

where $E_i=p_{\mu(i)}V^{\mu}_{(i)}$ is the energy of light ray relative to an observer O_i. Combining (2) and (19), we may write the ratio $(\lambda_{i+1}/\lambda_i)$ in terms of energy of photon and the world-function

$$\frac{\lambda_{i+1}}{\lambda_i}=\frac{l_{i+1}}{l_i}=\frac{E_i}{E_{i+1}}=\frac{p_{\mu(i)}V^{\mu}_{(i)}}{p_{\mu(i+1)}V^{\mu}_{(i+1)}}=\frac{\Omega_{\mu(i)}V^{\mu}_{(i)}}{\Omega_{\mu(i+1)}V^{\mu}_{(i+1)}}, \tag{20}$$

where $\Omega_{\mu(i)}=(u_{Oi}-u_S)U_{\mu}$. Therefore, the infinitesimal \`relative´ spectral shift δz_i between the observers O_{i+1} and O_i will be

$$\begin{aligned}\delta z_i=\frac{\delta\lambda_i}{\lambda_i}=\frac{\lambda_{i+1}-\lambda_i}{\lambda_i}=\frac{\delta l_i}{l_i}=\frac{l_{i+1}-l_i}{l_i}=\frac{p_{\mu(i)}V^{\mu}_{(i)}}{p_{\mu(i+1)}V^{\mu}_{(i+1)}}-1=\\ \frac{\Omega_{\mu(i)}V^{\mu}_{(i)}}{\Omega_{\mu(i+1)}V^{\mu}_{(i+1)}}-1=\frac{\tilde{\delta}z_i}{1+z_i}\equiv\frac{z_{i+1}-z_i}{1+z_i}.\end{aligned} \tag{21}$$

For definiteness, let consider case of $l_n>l_0$ (being red-shift, Fig. 1). In similar way, of course, we may treat a negative case of $l_n<l_0$ (being blue-shift), but it goes without saying that in this case a source is moving towards the observer. In first case, the observers at the points $O_{i(2)}$ $(i=1,...,n-1)$ should observe the monotonic increments of \`relative´ spectral shifts $(\delta z_1,\delta z_2,\delta z_3,...,\delta z_{n-1})$ when light ray passes across the infinitesimal distances $(O_{1(2)},S_{(2)}),(O_{2(2)},O_{1(2)}),...,(O_{n(2)},O_{(n-1)(2)})$. Thus, the wavelength of light emitted at $S_{(2)}$ is stretched out observed at the points $O_{i(2)}$. While weak, such effects considered cumulatively over a great number of successive increments of \`relative´ spectral shifts could become significant. The resulting spectral shift is the accumulation of a series of infinitesimal shifts as the light ray passes from luminous source to adjacent observers along the path of light ray. This interpretation holds rigorously even for large spectral shifts of order one or more. If this view would prove to be true, then it would lead to the chain rule for the wavelengths:

$$\frac{\lambda_{O_{(n2)}}}{\lambda_0}\equiv\frac{\lambda_n}{\lambda_1}=\frac{\lambda_n}{\lambda_{n-1}}\cdot\frac{\lambda_{n-1}}{\lambda_{n-2}}\cdots\frac{\lambda_2}{\lambda_1}\cdot\frac{\lambda_1}{\lambda_0}=\prod_{i=1}^{n-1}(1+\delta z_i), \tag{22}$$

where $\lambda_0\equiv\lambda_{S_{(2)}}$, such that

$$1+z=\frac{\lambda_n-\lambda_0}{\lambda_0}=\prod_{i=1}^{n-1}(1+\delta z_i)=\prod_{i=1}^{n-1}\frac{p_{\mu(i)}V^{\mu}_{(i)}}{p_{\mu(i+1)}V^{\mu}_{(i+1)}}=\prod_{i=1}^{n-1}\frac{\Omega_{\mu(i)}V^{\mu}_{(i)}}{\Omega_{\mu(i+1)}V^{\mu}_{(i+1)}}. \tag{23}$$

With no loss of generality, we may of course apply (23) all the way to $n\to\infty$. Let us view the increment of the proper space scale factor, $l_i=l(u_i)$, over the affine parameters u_i $(i=1,2,...,n)$ as follows: $l_i=l_0+i\varepsilon$, where ε can be made arbitrarily small by increasing n. In the limit $n\to\infty$, all the respective adjacent observers are arbitrarily close to each other, so that $\delta z_i=\delta l_i/l_i\simeq\varepsilon/l_0\to 0$. This allows us to write the following relation for the infinitesimal \`relative´ redshifts:

$$\begin{aligned}&(\delta z_{n-1}=\delta z_{n-2}=\cdots=\delta z_1=\varepsilon/l_0)_{n\to\infty}=\lim_{n\to\infty}\delta z^{(a)}_{(n-1)}\equiv\\ &\lim_{n\to\infty}\left(\frac{1}{n-1}\sum_{i=1}^{n-1}\delta z_i\right),\end{aligned} \tag{24}$$

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provided, $\delta z^{(a)}_{(n-1)}$ is the average infinitesimal increment of spectral shift. There does not seem to be any reason to doubt a validity of (24). Certainly, the identification adopted here can be readily proved as follows. The relation (23) then becomes

$$1+z=\lim_{n\to\infty}\prod_{i=1}^{n-1}(1+\delta z_i)=\lim_{n\to\infty}\left(1+\delta z^{(a)}_{(n-1)}\right)^{n-1}=\lim_{n\to\infty}(1+ \\ \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{\delta l_i}{l_i}\Big)^{n-1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\ln\frac{l_n}{l_0}\right)^{n}=\frac{l_n}{l_0}=\frac{p_{\mu(S)}V^{\mu}_{(S)}}{p_{\mu(O)}V^{\mu}_{(O)}}, \tag{25}$$

and hence

$$1+z=\frac{\triangle\tau_O}{\triangle\tau_S}=\frac{p_{\mu(S)}V^{\mu}_{(S)}}{p_{\mu(O)}V^{\mu}_{(O)}}=\frac{\Omega_{\mu(S)}V^{\mu}_{(S)}}{\Omega_{\mu(O)}V^{\mu}_{(O)}}= \\ 1+\sum_{i=1}^{n-1}\tilde{\delta}z_i=\lim_{n\to\infty}\prod_{i=1}^{n-1}(1+\delta z_i)=\lim_{n\to\infty}\left(1+\delta z^{(a)}_{(n-1)}\right)^{n-1}, \tag{26}$$

where $\Omega_{\mu(O)}=(u_O-u_S)U_{\mu(O)}$ and $\Omega_{\mu(S)}=-(u_O-u_S)U_{\mu(S)}$. The first line of equation (26) is the overall spectral shift rule (12), which proved a validity of the relation (24).

It is worth emphasizing that the general equation (26) is the result of a series of infinitesimal stretching of the proper space scale factor in Riemannian space-time, whereas the path of a luminous source appears nowhere, thus this equation does not relate to the special choice of transport path. Therefore, to overcome the ambiguity of parallel transport of four-velocities in curved space-time, in what follows we advocate exclusively with this proposal. To obtain some feeling about this statement, below we give more detailed explanation. The infinitesimal increments of `relative´ spectral shifts $(\delta z_1,\delta z_2,\delta z_3,...,\delta z_{n-1})$, according to (15), can be derived from Doppler effect between adjacent emitter and absorber in relative motion measured in the respective tangent local inertial rest frames at infinitesimally separated space-time points. To obtain some feeling about this statement, below we give more detailed explanation. Imagine a family of adjacent observers $(O_{a_i}(u_i))$ situated at the points a_i $(i=1,...,n)$ on the world lines (o_i) at infinitesimal distances from the observers $(O_{i(2)})$, who measure the wavelength of radiation in relative motion which cause a series of infinitesimal stretching $(\delta l_0,...,\delta l_{n-1})$ of the proper space scale factor. Since each line segment l_{i-1} of proper space scale factor (at the affine parameter u_{i-1}) is identically mapped on the line segment $(l_i-\delta l_{i-1})$ (where δl_i denotes infinitesimal segment $a_iO_{i(2)}$) of proper space scale factor (at the affine parameter $u_i=u_{i-1}+\delta u_{i-1}$), the relative speed $v_{O_{i(2)}O_{a_i}}(u_i)$ of observer $(O_{i(2)}(u_i))$ to adjacent observer $(O_{a_i})(u_i)$ should be the same as it is relative to observer $(O_{(i-1)(2)}(u_{i-1}))$, that is $v_{O_{i(2)}O_{(i-1)}}(u_{i-1})\equiv v_{O_{i(2)}O_{a_i}}(u_i)$. Continuing along this line, we may commit ourselves in the series of `relative´ spectral shifts, equivalently, a certain substitution of increments of relative speeds. Taking into account that the infinitesimal speeds of source (S) relative to observers $(O_{i(2)})$ arise at a series of infinitesimal stretching of the proper space scale factor δl_i $(i=1,2,...,n)$ as it is seen from the Fig. 1, we may fill out the whole pattern of monotonic increments of `relative´ spectral shifts $(\delta z_1,\delta z_2,\delta z_3,...,\delta z_{n-1})$ by, equivalently, replacing the respective pairs $(O_{1(2)},S_{(2)}),(O_{2(2)},O_{1(2)}),...,(O_{n(2)},O_{(n-1)(2)})$ with new ones $(O_{1(2)},O_{a_1}),(O_{2(2)},O_{a_2}),...,(O_{n(2)},O_{a_n})$, which attribute to the successive increments of relative speeds $v_{O_{1(2)}S}(u_1),...,v_{O_{n(2)}O_{(n-1)(2)}}(u_n)$ of the source (S) away from an observer $(O_{n(2)})$ in the rest frame of $(O_{n(2)})$, viewed over all the values $(i=1,...,n)$. This framework furnishes justification for the concept of relative speed $c\beta_n\equiv v_{O_{n(2)}S_{(2)}}$, to be now referred to as the *kinetic* relative velocity, of the source (S) to observer $(O_{n(2)})$ along the line of sight. According to relation (24), at the limit $n\to\infty$, the relative infinitesimal speeds tends to zero, $v_{O_{i(2)}O_{a_i}}(u_i)=c\delta\beta_i=c\delta z_i=c\delta l_i/l_i\simeq c\varepsilon/l_0\to 0$, such that

$$\lim_{n\to\infty}\delta\beta_1=\lim_{n\to\infty}\delta\beta_2=\cdots=\lim_{n\to\infty}\delta\beta_{n-1}= \\ \delta\beta^{(a)}\left(\equiv\lim_{n\to\infty}\frac{1}{n-1}\sum_{i=1}^{n-1}\delta\beta_i\right)=\lim_{n\to\infty}\frac{1}{n}\beta_n. \tag{27}$$

Remark: Although we are free to deal with any infinitesimal `relative´ spectral shift δz_i for the pair $(O_{i(2)})$ and (O_{a_i}), in local tangent inertial rest frame of an adjacent observer (O_{a_i}), where we may approximate away the curvature of space in the infinitesimally small neighborhood, nevertheless, the infinitesimal relative velocities arise in a generic pseudo-Riemannian space-time at a series of infinitesimal stretching of the proper space scale factor as alluded to above, so that the *SR law of composition*

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of velocities cannot be implemented globally along non-null geodesic because these velocities are velocities at the different events, which should be in a different physical frames, and cannot be added together.

Facilitating further the calculations of recession velocity (β_n) in quest, therefore, we may address the pair of observers at points $O_{(n)2}$ and a_n. Suppose $V^\mu_{O_{n(2)}}$ and $V^\mu_{O_{a_n}}$ be the unit tangent four-velocity vectors of observers ($O_{n(2)}$) and (O_{a_n}) to the respective world-lines in a general Riemannian space-time, thus in their respective rest frame we have $V^0_{O_{n(2)}} = 1$ and $V^0_{O_{a_n}} = 1$, as the only nonzero components of velocity. For comparing the vectors $V^\mu_{O_{n(2)}}$ and $V^\mu_{O_{a_n}}$ at different events, it is necessary to seek a useful definition of the relative velocity by bringing both vectors to a common event by subjecting one of them to parallel transport. Since all the paths between infinitesimally separated space-time points $O_{(n)2}$ and a_n are coincident at $n \to \infty$, for comparing these velocities there is no need to worry about specific choice of the path of parallel transport of four-vector. Therefore, we are free to subject further the unit tangent four-velocity vector $V^\mu_{O_{n(2)}}$ to parallel transport along the null geodesic $\Gamma_{O_{n(2)a_n}}$ to the point a_n. Thereby, the ray passes an observer $O_{a_n}(u_n)(\equiv O_{(n-1)(2)}(u_{n-1}))$ with the proper space scale factor l_{n-1} who measures the wavelength to be λ_{n-1}. The ray passes next observer $O_{n(2)}(u_n)$ with the proper space scale factor $l_n = l_{n-1} + \delta l_{n-1}$. The ray's wavelength measured by observer $O_{n(2)}(u_n)$ is increased by $\delta\lambda_{n-1} = \lambda_n - \lambda_{n-1}$ leading to infinitesimal `relative´ spectral shift δz_{n-1}. A parallel transport yields at O_{a_n} the vector $\beta_{\mu(O_{a_n})} = \mathrm{g}_{\mu\nu'}(O_{a_n}, O_{(n)2})V^{\nu'}_{O_{(n)2}}$, where the two point tensor $\mathrm{g}_{\mu\nu'}(O_{a_n}, O_{(n)2})$ is the parallel propagator as before, which is now determined by the points $O_{(n)2}$ and O_{a_n}. At $O_{(n)2} \to O_{a_n}$, we have the coincidence limit $[\mathrm{g}_{\mu\nu}](O_{a_n}) = g_{\mu\nu}(O_{a_n})$. As we have at point O_{a_n} two velocities $V^\mu_{O_{a_n}}$ and $\beta^\mu_{(O_{a_n})} = g^{\mu\nu}\beta_{\nu(O_{a_n})}$, according to the equations (14) and (15), we may associate Doppler shift δz_{n-1} to four-velocity $\beta^\mu_{(O_{a_n})}$ of observer O_n observed by an observer O_{a_n} with four-velocity $V^\mu_{O_{a_n}}$ as measured by the latter. Then, according to (16), the infinitesimal Doppler shift can be written:

$$\delta z_{n-1} = \frac{\delta\lambda_{n-1}}{\lambda_{n-1}} = 1 - \frac{1}{(1+\beta^2_{(O_{a_n})})^{1/2} + \beta_{R(O_{a_n})}} = \frac{p_{\mu(O_{a_n})}\beta^\mu_{(O_{a_n})}}{p_{\mu(O_{a_n})}V^\mu_{(O_{a_n})}} - 1, \tag{28}$$

where the notation of (16) is used but applied now to point (O_{a_n}) (instead of (O_1)). Consequently, the three-velocity of an observer ($O_{n(2)}$) relative to observer at (O_{a_n}) is $v_{(\alpha)(O_{a_n})}$, the relative speed is $v_{(O_{a_n})}$, and $v_{R(O_{a_n})}$ is the speed of recession of (o_n). In the local inertial rest frame $\xi^\mu_{(\alpha)(O_{a_n})}$ of an observer (O_{a_n}), the velocity vector $\beta^\mu_{O_{a_n}}$ takes the form $(\gamma, \gamma\delta\beta_{(O_{a_n})}, 00)$, where an observer ($O_n(2)$) is moving away from the observer (O_{a_n}) with the relative infinitesimal three-velocity $\delta\beta_{(O_{a_n})}$ (in units of the speed of light) in a direction making an angel $\theta_{(O_{a_n})}$ with the outward direction of line of sight $\Gamma_{O_{n(2)O_{a_n}}}$ from $O_{(n)2}$ to O_{a_n}, and $\gamma = (1 - \delta\beta^2_{(O_{a_n})})^{-1/2}$. Hence, the equation (28) is reduced to

$$\begin{aligned} \delta z_{n-1} = \frac{1+\delta\beta_{(O_{a_n})}\cos\theta_{(O_{a_n})}}{\sqrt{1-\delta\beta^2_{(O_{a_n})}}} - 1 = \beta_{R(O_{a_n})} - \beta^2_{R(O_{a_n})} + \tfrac{1}{2}\beta^2_{(O_{a_n})} + \cdots \simeq \beta_{R(O_{a_n})} = \\ \frac{p_{(\alpha)(O_{a_n})}v^{(\alpha)}_{(O_{a_n})}}{E_{O_{a_n}}} = \delta\beta_{(O_{a_n})}\cos\theta_{(O_{a_n})}. \end{aligned} \tag{29}$$

Thus, at $n \to \infty$, the wavelength measured by the observer $O_{n(2)}$ is increased by the first-order Doppler shift caused unambiguously by the infinitesimal relative speed $\delta\beta^{(r)}_{n-1} \equiv \delta\beta_{(O_{a_n})}\cos\theta_{(O_{a_n})}$ along the line of sight with end-points $O_{(n)2}$ and O_{a_n}:

$$\delta z_{n-1} = \frac{\delta l_{n-1}}{l_{n-1}} = \delta\beta^{(r)}_{n-1}. \tag{30}$$

The SR law of composition of velocities along the line of sight can be implemented in the tangent inertial rest frame of an observer O_{a_n}:

$$\delta\beta^{(r)}_{n-1} = \frac{\beta_n - \beta_{n-1}}{1-\beta_n\beta_{n-1}} \simeq \frac{\delta\beta_{n-1}}{1-\beta^2_{n-1}}, \tag{31}$$

where $v_{n-1} = c\beta_{n-1}$ and $v_n = c\beta_n$ are, respectively, the three-velocities of observers O_{a_n} and O_n along the line of sight with end-points O_{a_n} and $O_{(n)2}$. According to (27), at $n \to \infty$, a resulting infinitesimal

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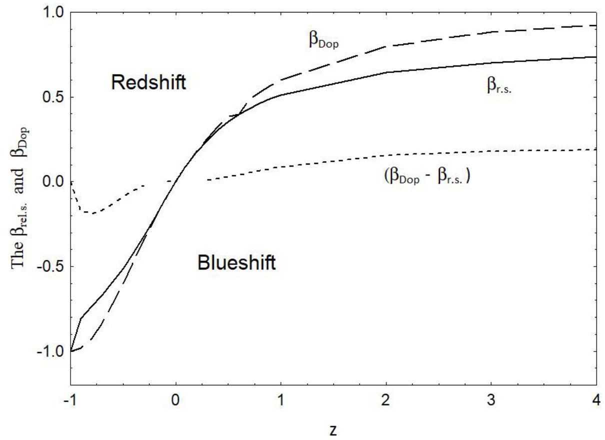

Figure 2. The relative velocity along the line of sight ($\beta_{r.s.}$) of luminous source (S) with $-1 \leq z \leq 4$ to observer (O), the global Doppler velocity (β_{Dop}), and their difference (in units of the speed of light).

increment δz_{n-1} of spectral shift reads

$$\lim_{n\to\infty}\delta z_{n-1} = \lim_{n\to\infty}\frac{\delta\beta_{n-1}}{1-\beta^2_{n-1}} = \lim_{n\to\infty}\frac{\beta_n}{n(1-\beta^2_n)}, \tag{32}$$

For our goal, the most straightforward guess at the convenient form of (26), by virtue of (24), is given by

$$1+z = \frac{p_{\mu S}V^\mu_S}{p_{\mu O}V^\mu_O} = \frac{\Omega_{\mu(S)}V^\mu_{(S)}}{\Omega_{\mu(O)}V^\mu_{(O)}} = \lim_{n\to\infty}(1+\delta z_{n-1})^n. \tag{33}$$

Certainly, it is more rewarding to go ahead with this relation, which incorporated with (32), yield a finite spectral shift

$$1+z = \frac{p_{\mu S}V^\mu_S}{p_{\mu O}V^\mu_O} = \frac{\Omega_{\mu(S)}V^\mu_{(S)}}{\Omega_{\mu(O)}V^\mu_{(O)}} = \lim_{n\to\infty}\left[1+\frac{1}{n}\left(\frac{\beta_n}{(1-\beta^2_n)}\right)\right]^n. \tag{34}$$

This straightforwardly leads to the kinematic relationship of the overall spectral shift, z, and the speed $\beta_{r.s.}$ (in units of the speed of light) of source (S) relative to observer ($O \equiv O_n$) in its rest frame along the line of sight in a general Riemannian space-time:

$$1+z = \frac{p_{\mu S}V^\mu_S}{p_{\mu O}V^\mu_O} = \frac{\Omega_{\mu(S)}V^\mu_{(S)}}{\Omega_{\mu(O)}V^\mu_{(O)}} = \exp\left(\frac{\beta_{r.s.}}{1-\beta^2_{r.s.}}\right), \tag{35}$$

where, hereinafter, the relative speed $\beta_{r.s.} \equiv \lim_{n\to\infty}\beta_n$ in quest is marked with subscript $()_{r.s.}$. The relative speed of luminous source is plotted on the Fig. 2 for redshifts $-1 \leq z \leq 4$. Next we will study a particular case of establishing a global Doppler shift from a general solution (35).

2.1. A global Doppler shift along the null geodesic

Suppose the velocities of observers say $O_{i(2)}$ $(i = 1, ..., n-1)$, being in free fall, populated along the null geodesic $\Gamma_{S_{(2)}O_{(2)}}(v + \triangle v)$ of light ray (Fig. 1), vary smoothly along the line of sight with the infinitesimal increment of relative velocity $\delta\beta^r_i$. The (i)-th observer situated at the point $i(2)$ of intersection of the ray's trajectory $\Gamma_{S_{(2)}O_{(2)}}(v + \triangle v)$ with the world line (o_i) at affine parameter u_i, and measures the frequency of light ray as it goes by. According to the equivalence principle, we may approximate away the curvature of space in the infinitesimally small neighborhood of two adjacent observers. We should emphasize that if we approximate an infinitesimally small neighborhood of a curved space as flat, the resulting errors are of order $(\delta l_i/l_n)^2$ in the metric. If we regard such errors as negligible, then we can legitimately approximate space-time as flat. The infinitesimal increment of spectral shift δz_i is not approximated away in this limit because it is in that neighborhood of leading order $(\delta l_i/l_i)$. That is, approximating away the curvature of space in the infinitesimally small neighborhood does not mean approximating away the infinitesimal increment δz_i. Imagine a thin

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world tube around the null geodesic $\Gamma_{S_{(2)}O_{(2)}}(v+\triangle v)$ within which the space is flat to arbitrary precision. Each observer has a local reference frame in which SR can be taken to apply, and the observers are close enough together that each one $O_{(i+1)(2)}$ lies within the local frame of his neighbor $O_{i(2)}$. This implies the vacuum value of a velocity of light to be universal maximum attainable velocity of a material body found in this space. Such statement is true for any thin neighborhood around a null geodesic. Only in this particular case, the relative velocity of observers can be calculated by the SR law of composition of velocities globally along the path of light ray. We may apply this law to relate the velocity β_{i+1} to the velocity β_i, measured in the i-th adjacent observer's rest frame. The end points of infinitesimal distance between the adjacent observers $O_{(i+1)(2)}$ and $O_{i(2)}$ will respectively be the points of intersection of the ray's trajectory with the world lines $o_{i+1}(u_{i+1})$ and $(o_i)(u_i)$. This causes a series of infinitesimal increment of the proper space scale factor from initial value $l_0=\triangle\tau_S$ to the given value $l_i=\triangle\tau_{O_i}$, which in turn causes a series of infinitesimal increment of spectral shift $\delta z_i=\delta\lambda_i/\lambda_i=\delta l_i/l_i$. Within each local inertial frame, there are no gravitational effects, and hence the infinitesimal spectral shift from each observer to the next is a Doppler shift. Thus, at the limit $n\to\infty$, a resulting infinitesimal frequency shift δz_i, can be unambiguously equated to infinitesimal increment of a fractional SR Doppler shift $\delta\bar{z}_i$ from observer $O_{i(2)}$ to the next $O_{(i+1)(2)}$ caused by infinitesimal relative velocity $\delta\bar{\beta}^r_i$:

$$\left(\delta z_i=\frac{\delta l_i}{l_i}\right)_{n\to\infty}=\left(\delta\bar{z}_i=\delta\bar{\beta}^r_i=\frac{\bar{\beta}_{i+1}-\bar{\beta}_i}{1-\beta_{i+1}\beta_i}\simeq\frac{\delta\bar{\beta}_i}{1-\bar{\beta}_i^2}\right)_{n\to\infty}, \tag{36}$$

where by $\bar{()}$ we denote the null-geodesic value, as different choice of geodesics yields different results for the motion of distant test particles relative to a particular observer. The relation (36), incorporated with the identity (24), yield

$$(\delta z_{n-1}=)_{n\to\infty}=\left(\frac{\delta\beta_{n-1}}{1-\beta_n^2}\right)_{n\to\infty}=\left(\delta\bar{z}^{(a)}_{(n-1)}=\delta\bar{\beta}^{r(a)}_{(n-1)}\equiv \frac{1}{n-1}\sum_{i=1}^{n-1}\frac{\delta\bar{\beta}_i}{1-\bar{\beta}_i^2}\right)_{n\to\infty}, \tag{37}$$

which, by virtue of (27), for sufficiently large but finite n gives

$$\frac{\beta_n}{1-\beta_n^2}=\sum_{i=1}^{n-1}\frac{\delta\bar{\beta}_i}{1-\bar{\beta}_i^2}=\int_0^{\bar{\beta}_n}\frac{d\bar{\beta}}{1-\bar{\beta}^2}, \tag{38}$$

or

$$\bar{\beta}_n=\frac{e^{\varrho_n}-1}{e^{\varrho_n}+1},\quad \varrho_n\equiv\frac{2\beta_n}{1-\beta_n^2}. \tag{39}$$

Hence the general solution (35), by means of relation (38), is reduced to a global Doppler shift (14) along the null geodesic:

$$1+z=\exp\left(\frac{\beta_{r.s.}}{1-\beta^2_{r.s.}}\right)=\sqrt{\frac{1+\bar{\beta}_{r.s.}}{1-\bar{\beta}_{r.s.}}}=\frac{p_{\mu(O2)}V^{\mu}_{(S2)}}{p_{\mu(O2)}V^{\mu}_{(O2)}}, \tag{40}$$

where $\bar{\beta}_{rec}=\lim_{n\to\infty}\bar{\beta}_n$, $V^{\mu}_{(S2)}$ and $V^{\mu}_{(O2)}$ are the four-velocity vectors, respectively, of the source $S_{(2)}$ and observer $O_{(2)}$, $p_{\mu(S2)}$ and $p_{\mu(O2)}$ are the tangent vectors to the typical null geodesics $\Gamma_{S_{(2)}O_{(2)}}(v)$ at their respective end points. This procedure, in fact, is equivalent to performing parallel transport of the source four-velocity in a general Riemannian space-time along the null geodesic to the observer. Note that any null geodesic from a set of null geodesics mapped (s) on (o) can be treated in the similar way. In Minkowski space a parallel transport of vectors is trivial and mostly not mentioned at all. This allows us to apply globally the SR law of composition of velocities to relate the velocities $\bar{\beta}_{i+1}$ to the $\bar{\beta}_i$ of adjacent observers along the path of light ray, measured in the i-th adjacent observer's frame. Then, according to (36)-(40), a global Doppler shift of light ray emitted by luminous source as it appears to observer at rest in flat Minkowski space can be derived by summing up the infinitesimal Doppler shifts caused by infinitesimal relative velocities of adjacent observers.

2.2. The implications for the standard cosmological model

In the framework of standard cosmological model, one assumes that the universe is populated with comoving observers. In the homogeneous, isotropic universe comoving observers are in freefall, and

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obey Wayl's postulate: their all worldlines form a 3-bundle of non-intersecting geodesics orthogonal to a series of spacelike hypersurfaces, called comoving hypersurfaces. In case of expansion, all worldlines are intersecting only at one singular point. The clocks of comoving observers, therefore, can be synchronized once and for all. Let the proper time, t, of comoving observers be the temporal measure. Suppose $R(t)$ is the scale factor in expanding homogenous and isotropic universe. One considers in, so-called, cosmological rest frame a light that travels from a galaxy to a distant observer, both of whom are at rest in comoving coordinates. As the universe expands, the wavelengths of light rays are stretched out in proportion to the distance $L(t)$ between co-moving points ($t > t_1$), which in turn increase proportionally to $R(t)$ (Harrison, 1993, 1995):

$$\frac{\lambda(t)}{\lambda(t_1)} = \frac{dt}{dt_1} = \frac{R(t)}{R(t_1)} = \frac{L(t)}{L(t_1)}. \tag{41}$$

Reviewing notations in this `cosmic wavelength stretching´ relation, $L_1 \equiv L(t_1)$ is the proper distance to the source at the time when it emits light, $L(t)$ is the same distance to the same source at light reception.

In what follows, the mathematical structure has much in common with those constructions used for deriving of (21)-(35). After making due allowances for (41), particularly, the infinitesimal `relative´ increment δz_j ($j = 1, ..., n-1$) of redshift reads

$$\delta z_j = \frac{\delta\lambda_j}{\lambda_j} = \frac{\lambda_{j+1}-\lambda_j}{\lambda_j} = \frac{\delta L_j}{L_j} = \frac{L_{j+1}-L_j}{L_j} = \frac{\tilde{\delta} z_j}{1+z_j} \equiv \frac{z_{j+1}-z_j}{1+z_j}, \quad 1+z_j = \frac{\delta\lambda_j}{\lambda_1}, \tag{42}$$

where, the role of proper space scale factor l_i is now destined to the scale factor $R(t_i) \propto L(t_i)$. Consequently, the general relation (35) straightforwardly yields the particular solution as a corollary for the case of expanding RW space-time of standard cosmological model, i.e. the kinematic relationship of the overall cosmological redshift, z, and *kinetic* recession velocity ($\beta_{rec} \equiv \beta_{r.s.}$) (in units of the speed of light) of the comoving distant galaxy (A_1) of redshift z, which crossed past light cone at time t_1 away from comoving observer (O):

$$1 + z = \frac{R(t)}{R(t_1)} = \exp\left(\frac{\beta_{rec}}{1-\beta_{rec}^2}\right). \tag{43}$$

This interpretation so achieved has physical significance as it agrees with a view that the light waves will be stretched by travelling through the expanding universe, and in the same time the *kinetic* recession velocity of a distant astronomical object is always subluminal even for large redshifts of order one or more. It, therefore, does not violate the fundamental physical principle of *causality*. Moreover, the general solution is reduced to global Doppler shift (40) along the null geodesic, studied by Synge (1960) (see also Bunn & Hogg (2009), Narlikar (1994)).

If, and only if, for the distances at which the Hubble empirical linear `redshift-distance´ law ($cz = HL$) is valid, the relationship between the *physical* recession velocity, v_{rec}, and the expansion rate, $\dot{L}$ ($= HL$), reads

$$\beta_{rec} = \frac{\sqrt{1+4\ln^2(1+\dot{L}/c)}-1}{2\ln(1+\dot{L}/c)}. \tag{44}$$

3. Concluding remarks

Let us briefly summarize the main results of this work. This report is about the much-discussed in literature question of interpretation of the spectral shift of radiation from a distant object in a curved spacetime. We aim to provide a unique definition for the *kinetic* relative velocity between a source and the observer as measured along the observer's line-of-sight. Extending those geometrical ideas of well-known kinematic spectral shift rule to infinitesimal domain, we try to catch this effect by building a series of infinitesimally displaced shifts and then sum over them in order to find the proper answer to the problem that we wish to address. Thereby, the general equation (26) is the result of a series of infinitesimal stretching of the proper space scale factor in Riemannian space-time, whereas the path of a luminous source appears nowhere, thus this equation does not relate to the

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special choice of transport path. A resulting general relationship (35) between the spectral shift and the *kinetic* relative velocity is utterly distinct from a familiar global Doppler shift (14). We discuss the implications for a particular case when adjacent observers are being in free fall and populated along the null geodesic, so that the *kinetic* relative velocity of luminous source is reduced to global Doppler velocity (40) as studied by Synge. Moreover, the implications for the spatially homogeneous and isotropic RW space-time of standard cosmological model leads to cosmological consequences that resulting *kinetic* recession velocity of a distant astronomical object is always subluminal even for large redshifts of order one or more and, thus, it does not violate the fundamental physical principle of *causality*.

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