# Relative velocity in pseudo-Riemannian spacetime

G.Ter-Kazarian*

Byurakan Astrophysical Observatory, 378433, Aragatsotn District, Armenia

**Abstract**

We give a coordinate independent definition of relative velocity of test particle in pseudo-Riemannian spacetime as measured along the observer's line-of-sight in general and several instructive cases. In doing this, the test particle is considered as a luminous object, otherwise, if it is not, we assume that a light source is attached to it, which has neither mass nor volume. Then we utilize the general solution of independent definition of relative velocity of a luminous source in generic pseudo-Riemannian spacetime. As a corollary, we discuss the implications for the Minkowski metric, the test particle and observer at rest in an arbitrary stationary metric, the uniform gravitational field, the rotating reference frame, the Schwarzschild metric, the Kerr-type metrics, and the spatially homogeneous and isotropic Robertson-Walker (RW) spacetime of standard cosmological model. In the last case, it leads to cosmological consequence that the resulting, so-called, *kinetic* recession velocity of an astronomical object is always subluminal even for large redshifts of order one or more, so that it does not violate the fundamental physical principle of *causality*. We also calculate the measure of carrying away of a galaxy at redshift $z$ by the expansion of space, which proves, in particular, that cosmological expansion of a flat 3D–space is fundamentally different from a kinematics of galaxies moving in a non-expanding flat 3D-space. So, it is impossible to mimic the true cosmological redshift by a Doppler effect caused by motion of galaxies in a non-expanding 3D-space, flat or curved. We also give a reappraisal of the `standard´ kinematic interpretation of redshifts in RW spacetime as accumulated Doppler-shifts.

***Keywords:*** *Classical general relativity; Fundamental problems and general formalism; Riemannian geometries; Relative velocity; Classical black holes*

## 1. Introduction

This article is an extended version of (Ter-Kazarian, 2023), where we have studied the question of a coordinate independent relative velocity in pseudo-Riemannian spacetime. Here, in addition to more detailed exposition, we also include some relevant topics. In particular, Appendix A provides a reappraisal of the `standard´ kinematic interpretation of redshifts in RW spacetime as accumulated Doppler-shifts.

The relative velocity of test particles and observers, which is fundamental notion in physics, has been an open question in general relativity (GR). From almost its very outset there has been an ongoing quest for determining the relative velocity of test particles and observers, which does not depend on coordinates. There is no unique way to compare the four-vectors of their velocities at widely separated spacetime events in curved pseudo-Riemannian spacetime, because GR provides no *a priori* definition of relative velocity. This inability to compare vectors at different points was the fundamental feature of a curved spacetime. Different coordinate reference frames and notions of relative velocity yield different results for the motion of distant test particles relative to a particular observer.

Perhaps the most convincing example of this is Milne's universe (Milne, 1934). Against the background of empty Minkowski space $M_4$ and complete disregard for gravity, Milne considered an infinite number of test particles (no mass, no volume) shot out in all directions and with all possible speeds, in a unique event of creation occurred in his origin point $O$ at $T = 0$. All the particles, being free,

*gago_50@yahoo.com


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will move uniformly and radially away from $O$, with all possible speeds short of $c$. The particles in this idealized model each stay at fixed comoving coordinates $\chi(x, y, z)$ as their proper time $t$ increases. The physical distance between two particles, as measured along a geodesic of a comoving hypersurface of constant $t$, grows at the Hubble expansion rate of the universe at the time $t$. So the Milne universe is the Minkowski spacetime described from an expanding reference frame. The picture in some particular inertial frame $S(X, Y, Z, T)$ of arbitrarily chosen reference particle $P$ will be that of a ball of dust whose unattained boundary expands at the speed of light. Milne's model satisfies the Cosmological Principle of homogeneity and isotropy. The Minkowski coordinates are the coordinates of a rigid inertial reference frame of the expanding cloud of particles defining the Milne universe model. The time $T$ is the private time of comoving inertial reference frame of an arbitrary particle $P$ in the Milne universe. All the clocks at rest in this frame show the same time $T$ as the clock at the spatial origin of the reference frame. The coordinate $R$ is the distance from the origin particle measured at $T$=const. The fan of lines is the set of world lines of the reference particles defining the Milne universe. The cosmic time $t$ is the time measured by clocks following all of the test particles. The space at a fixed cosmic time $t$ is called the public space of the Milne universe. Each public map in Milne's model is a 3-space of constant negative curvature. The coordinate transformation between the Minkowski coordinates $(R(X, Y, Z), T)$ and the Milne coordinates $(t, \chi)$, is $R = ct \sinh(\chi/ct_0)$, $T = t \cosh\cosh(\chi/ct_0)$, which leads to the line element $ds^2 = c^2dt^2 - d\sigma^2$, where $d\sigma^2 = a^2(t)dl^2$ is the metric of public space with scale factor $a(t) = t/t_0$. By Schur's theorem, $dl^2$ represents a 3D-space of constant curvature. For the spatially flat model, there is no upper limit to the comoving coordinate distance $\chi$, and that also to the proper distance $L(t) = a(t)\chi$, at any fixed $t$. For the Milne universe, at any time $t$ and at sufficiently large proper distance, the `proper´ recession velocity $\dot{L}(t, z) = HL$, where $h = \dot{a}(t)/a(t)$ is the Habble parameter necessarily exceeds the local speed of light for the observer. Although in the Milne coordinates the 3D comoving hypersurface of constant $t$ (constant proper time) does have an extrinsically non-zero curvature, nevertheless the 4D curvature is zero. Therefore, simple coordinate transformations to the Minkowski coordinates as above, transform the metric to the standard Minkowski form. In these coordinates there are no superluminal speeds.

The ambiguity illustrated by this example has analogs in all spacetimes. This feature led to consideration of the need for a strict definition of `radial velocity´ within the solar system at the General Assembly of the International Astronomical Union (IAU), held in 2000 (Lindegren & Dravins, 2003, Soffel, 2003). Whereas, the metric tensors and gravitational potentials of both the Barycentric Celestial Reference System and the Geocentric Celestial Reference System are defined and discussed. The necessity and relevance of the two celestial reference systems are explained. The transformations of coordinates and gravitational potentials are discussed. Potential coefficients parameterizing the post-Newtonian gravitational potentials are expounded. Simplified versions of the time transformations suitable for modern clock accuracies are elucidated.

Several papers appeared that study the general question of relative velocities. The result of such efforts was the introduction of four geometrically defined inequivalent concepts of relative velocity. The three distinct coordinate charts are employed by Bolós (2006, 2007), Bolós & Klein (2012), Bolós et al. (2002), Klein & Collas (2010), Klein & Randles (2011), each with different notions of simultaneity, to calculate the Fermi, kinematic, astrometric, and spectroscopic relative velocities. The four definitions of relative velocities depend on two different notions of simultaneity: `spacelike simultaneity´ (or `Fermi simultaneity´) (Klein & Randles, 2011, Walker, 1935) as defined by Fermi coordinates of an observer, and `lightlike simultaneity´ as defined by optical (or observational) coordinates of an observer (Ellis, 1985). The Fermi and kinematic relative velocities can be described in terms of the `Fermi simultaneity´, according to which events are simultaneous if they lie on the same space slice determined by Fermi coordinates. Thereby, for an observer following a timelike worldline in Riemannian spacetime, Fermi-Walker coordinates provide a system of locally inertial coordinates. If the worldline is geodesic, the coordinates are commonly referred to as Fermi or Fermi normal coordinates. Useful feature of Fermi coordinates was that the metric tensor expressed in these coordinates is Minkowskian to first order near the path of the Fermi observer, with second order corrections involving only the curvature tensor (Manasse & Misner, 1963). Klein & Randles (2011) find explicit expressions for the Fermi coordinates for Robertson-Walker (RW) spacetimes and show that the Fermi


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chart for the Fermi observer in non-inflationary RW spacetimes is global. However, rigorous results for the radius of a tubular neighborhood of a timelike path for the domain of Fermi coordinates are not available. The spectroscopic (or barycentric) and astrometric relative velocities, which can be derived from spectroscopic and astronomical observations, mathematically, both rely on the notion of light cone simultaneity. According to the latter, two events are simultaneous if they both lie on the same past light cone of the central observer. It is shown that the astrometric relative velocity of a radially receding test particle cannot be superluminal in any expanding RW spacetime. Necessary and sufficient conditions are given for the existence of superluminal Fermi speeds. Note that for the Hubble velocity, the proper distance is measured along non geodesic paths, while for the Fermi velocity, the proper distance is measured along spacelike geodesics. In this respect the Fermi velocity seems to be more natural, but the Hubble velocity is defined at all spacetime points, whereas the Fermi velocity makes sense only on the Fermi chart of the central observer. Although these four definitions of relative velocities have own physical justifications, all they are subject to many uncertainties, and the ambiguity still remains.

In this article, we are looking for a solution by considering the test particle as a luminous object, otherwise, if it is not, we assume that a light source is attached to it, which has neither mass nor volume. In such case, a hope appears that a relative velocity of luminous source as measured along the observer's line-of-sight (speed) can be defined in coordinate independent way directly from general kinematic spectral shift rule (Synge, 1960):

$$z = \frac{\triangle\,\tau_O}{\triangle\,\tau_S} - 1 = \frac{p_{\mu(S)}V^{\mu}_{(S)}}{p_{\mu(O)}V^{\mu}_{(O)}} - 1, \tag{1}$$

which is generally valid for pseudo-Riemannian manifold, and thus, the definitions will be free of the above mentioned ambiguities. Here $V^{\mu}_{(S)}$ and $V^{\mu}_{(O)}$ are respective unit tangent vectors of four-velocity vectors of source $S$ and observer $O$, $p^{\mu}_{(S)}$ and $p^{\mu}_{(O)}$ are respective four-momenta of light ray as seen by the source and observer. An aforementioned inability to immediately compare the four-velocity vector $V^{\mu}_{S}$ with the four-velocity vector $V^{\mu}_{O}$ in pseudo-Riemannian spacetime necessitates to seek a useful definition of the relative velocity by bringing both vectors to a common event. Historically, Synge subjected the vector $V^{\mu}_{(S)}$ to parallel transport along the null geodesic to the observer and obtained a relativistically invariant form of global Doppler shift. Narlikar (Narlikar, 1994) has proved this rule in other context of standard cosmological model of expanding universe.

It is well known that null geodesics are peculiar, in a sense that they lie in a metric space wherein they are being only 1D-affine spaces, so that only a parallel displacement, not a metric distance is defined along them. As a corollary, their geometric properties become a rather unexpected mixture of affine and metric properties. At first glance, we seem to have attractive proposal of choosing a null geodesics for the parallel transport since it does not require any additional structures, like particular foliation of spacetime, which in turn is applicable to any spacetime. However, a resulting Doppler effect is inconclusive, because a definition of relative velocity has disadvantage that there is no unique way to compare four-vectors of the velocities at widely separated spacetime events by parallel transport.

Our primary interest, in (Ter-Kazarian, 2021b) is rather to extend those geometrical ideas developed by (Synge, 1960), to build a series of infinitesimally displaced `relative´ spectral shifts and then sum over them in order to give a coordinate independent definition of the relative velocity of luminous source, without subjecting it to a parallel transport, as measured along the observer's line-of-sight in generic pseudo-Riemannian spacetime. These peculiarities deserve careful study, because they furnish valuable theoretical clues about the interpretation of velocity of test particle relative to observer in GR, a systematic analysis of properties of which happens to be surprisingly difficult by conventional methods. The parallel transport of four-velocity vector has not been discussed in this article, although the actual formulation of theory has made the task easier. A resulting general relationship between the spectral shift and relative velocity is utterly distinct from a familiar global Doppler shift. In particular case when adjacent observers are being in free fall and populated along the null geodesic, such a performance is reduced to a global Doppler velocity as studied by Synge (Synge, 1960).

Utilizing this general solution, in present article we discuss the implications for several instructive cases: the Minkowski metric, the test particle and observer at rest in an arbitrary stationary metric,


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the uniform gravitational field, the rotating reference frame, the Schwarzschild metric, the Kerr-type metrics, and the spatially homogeneous and isotropic Robertson-Walker spacetime of standard cosmological model. In the last case, the general solution gives a coordinate independent definition of *kinetic* recession velocity, so that, reconciles the cosmological interpretation of redshift with the kinematical interpretation of redshifts as accumulation of a series of `relative´ infinitesimal spectral shifts. Such a practical implementation leads to important cosmological consequence that the *kinetic* recession velocity is always subluminal even for large redshifts of order one or more (Ter-Kazarian, 2021a, 2022), and thus, it does not violate the fundamental physical principle of *causality*. This provides a new perspective to solve startling difficulties of superluminal `proper´ recession velocities, which the conventional scenario of expanding universe of standard cosmological model presents (see e.g. (Bolós & Klein, 2012, Bunn & Hogg, 2009, Chodorowski, 2011, Davis & Lineweaver, 2004, Grøn & Elgarøy, 2007, Harrison, 1993, 1995, 2000, Kaya, 2011, Klein & Collas, 2010, Klein & Randles, 2011, Murdoch, 1977, Narlikar, 1994, Page, 2009, Peacock, 1999, 2008, Peebles, 1993, Peebles et al., 1991, Whiting, 2004). In some instances (in earlier epochs), the distant astronomical objects are observed to exhibit redshifts in excess of unity, and only a consistent theory could tackle the key problems of a dynamics of such objects.

With this perspective in sight, we will proceed according to the following structure. To start with, section 2 deals with both the coordinate independent definition and calculation of relative velocity of test particle as measured along the observer's line-of-sight in generic pseudo-Riemannian spacetime. We defer a derivation of a Doppler velocity from a general solution of relative velocity of source along the null geodesic, as it is studied by (Synge, 1960), to the section 3. We use the general solution as a backdrop to explore in section 4 the special cases of the Minkowski metric, the test particle and observer at rest in an arbitrary stationary metric, the uniform gravitational field, the rotating reference frame, the Schwarzschild metric, the Kerr-type metrics, and the spatially homogeneous and isotropic Robertson-Walker spacetime of standard cosmological model. Concluding remarks are given in section 5. Appendix A provides a reappraisal of the `standard´ kinematic interpretation of redshifts in RW spacetime as accumulated Doppler-shifts. Unless otherwise stated, we are not using the $c = 1$ convention here, so the light velocity $c$ appears in all formulae where it should.

## 2. The relative velocity of luminous source in generic pseudo-Riemannian spacetime, revisited

For a benefit of the reader, in this section we necessarily revisit a general solution for relative velocity of luminous source, an early version of which is given in (Ter-Kazarian, 2021b). In this section much more will be done to make the early results and formulations clear and rigorous, which are in use throughout the present paper. Therein, we divide the spectral shift into infinitesimally shifted `relative´ spectral intervals, measured at infinitesimally separated points of spacetime between neighboring adjacent observers, and sum them over to remove the ambiguity that represents the parallel transport of the four-velocity of source to the observer. To clarify our setup, it should help a few noteworthy points of Fig. 1. The $(o)$ and $(s)$ are two world lines respectively of observer $O$ and source $S$ in pseudo-Riemannian spacetime. The passage of light signals from $S$ to $O$ is described by a single infinity of null geodesics $\Gamma(v)$ connecting their respective world lines. The $S_{(1)}$ and $S_{(2)}$ are two neighboring world points on $(s)$. The parametric values for these geodesics are $v, v+ \triangle v$, respectively, where $v = const$ and $\triangle v$ is infinitesimally small. Accordingly, the world line $(s)$ is mapped pointwise on the $(o)$ by a set of null geodesics $\Gamma(v)$. That is, a set of null geodesics are joining $(s)$ to $(o)$, each representing the history of a wave crest. The totality of these null geodesics forms a 2-space with equation $x^\mu = x^\mu(u, v)$, which is determined once $(s)$ and $(o)$ are given. The $u$ denote the affine parameter on each of these geodesics running between fixed end-values $u = 0$ on $(s)$ and $u = 1$ on $(o)$. The $O_{(1)}$ and $O_{(2)}$ are corresponding world points on $(o)$, where the null geodesics from $S_{(1)}$ and $S_{(2)}$ meet it. Also the proper times of the observer and the source are, respectively, denoted by $\tau_O$ and $\tau_S$, and $\triangle \tau_O$ and $\triangle \tau_S$ are the elements of proper time corresponding to the segments (the clock measures of) $O_{(1)}O_{(2)}$ and $S_{(1)}S_{(2)}$. A dense family of adjacent observers $O_j$ $(j = 1, ..., n-1)$ with the world lines $(o_j)$ are populated between the two world lines $(o)$ and $(s)$. Each observer $O_j$


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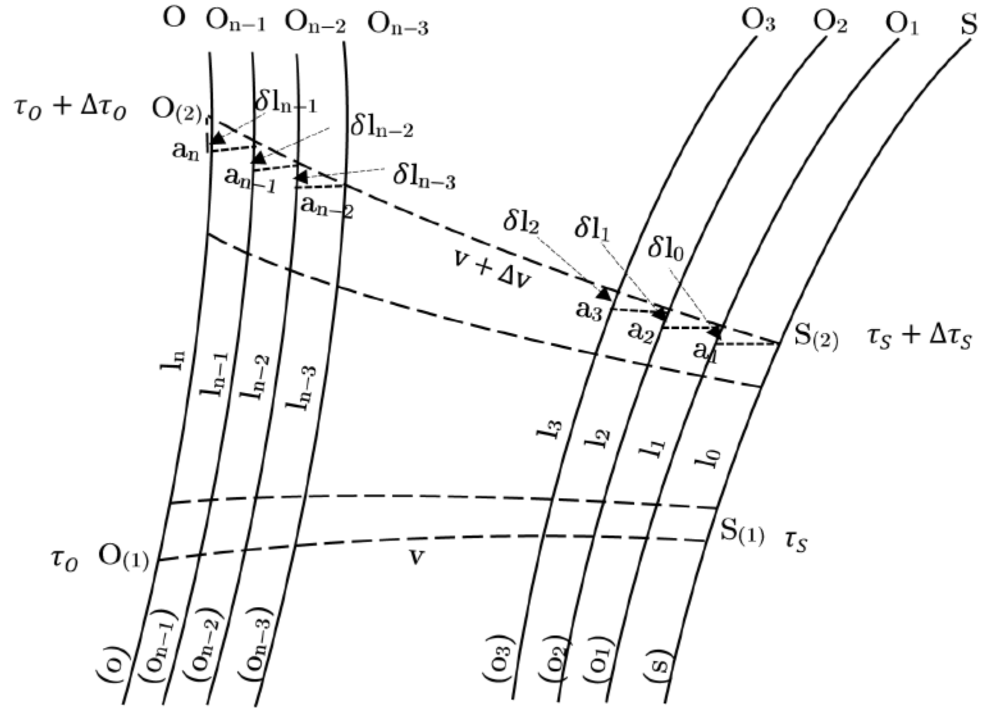


Figure 1. The infinitesimal spectral shifts as measured locally by emitter and adjacent receivers in generic pseudo-Riemannian spacetime. The $(o)$ and $(s)$ are two world lines respectively of observer $O$ and source $S$. A dense family of adjacent observers $O_j$ $(j = 1, ..., n-1)$ with the world lines $(o_j)$ populated between the two world lines $(o)$ and $(s)$. A set of null geodesics (the dotted lines) is mapping $(s)$ on $(o)$, each representing the history of a wave crest. Each line segment $l_{i-1}$ of proper space scale factor (at the affine parameter $u_{i-1}$) is identically mapped on the line segment $(l_i - \delta l_{i-1})$ of proper space scale factor (at infinitesimally close affine parameter $u_i$), such that $l_{i-1} \equiv (l_i - \delta l_{i-1})$, where $\delta l_i$ denotes infinitesimal segment $a_i O_{i(2)}$. The null geodesic light signals travelling from an adjacent observer $(O_{a_n})$, located at point $a_n$, to the observer $(O_{(2)})$, is also schematically depicted with large dashed line.

measures the frequency of light rays emitted by the source $S$ as it goes by. The $O_{j(1)}$ and $O_{j(2)}$ are two neighboring world points on $(o_j)$ where the null geodesics from $S_{(1)}$ and $S_{(2)}$ meet it. The $u_j$ denote the values of affine parameter on each of the null geodesics chosen at equal infinitesimally small $\delta u_j$, so that $u = u_j$ on $(o_j)$. The $\tau_{O_j}$ denotes the proper times of the adjacent observers, i.e. $\triangle\, \tau_{O_j}$ are the elements of proper time corresponding to the segment $O_{j(1)}O_{j(2)}$. Here and throughout, for a mere convenience, we use the proper space scale factor $l_i$ $(i = 0, 1, 2, ..., n)$ which encapsulates the beginning and evolution of the elements of proper time: $l_0 = c \triangle \tau_S$, $l_1 = c \triangle \tau_{O_{(1)}}, \cdots$ , $l_{n-1} = c \triangle \tau_{O_{n-1}}$, $l_n = c \triangle \tau_O$. Each line segment $l_{i-1}$ of proper space scale factor (at the affine parameter $u_{i-1}$) is identically mapped on the line segment $(l_i - \delta l_{i-1})$ of proper space scale factor (at infinitesimally close affine parameter $u_i$), such that $l_{i-1} \equiv (l_i - \delta l_{i-1})$, where $\delta l_i$ denotes infinitesimal segment $a_i O_{i(2)}$. The null geodesic light signals travelling from an adjacent observer $(O_{a_n})$, located at point $a_n$, to the observer $(O_{(2)})$, is also schematically depicted with large dashed line. The wavelength $\lambda_i$ of light ray is varied on the infinitesimal distance between the observers $O_{i+1}$ and $O_i$ in a general Riemannian spacetime in proportion to the proper space scale factor $l_i$ :

$$\frac{\lambda_{i+1}}{\lambda_i} = \frac{\triangle\, \tau_{O_{i+1}}}{\triangle\, \tau_{O_i}} = \frac{l_{i+1}}{l_i}, \tag{2}$$

such that the spectral shift $z_i$ is defined as the fractional shift in wavelength

$$z_i = \frac{\lambda_i}{\lambda_S} - 1 = \frac{l_i}{l_0} - 1 = \frac{\triangle\, \tau_{O_i}}{\triangle\, \tau_S} - 1. \tag{3}$$


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Following (Ter-Kazarian, 2021b), the spectral shift $z_i$, in general, can be evaluated straightforwardly in terms of the world function $\Omega(SO_i)$ for two points $S(x')$ and $O_i(x_{(i)})$ $(i = 1, ..., n)$ through an integral defined along the geodesic $\Gamma_{SO_i}(v)$ joining them (Synge, 1960), taken along any one of the curves $v = const$. The world function $\Omega(SO_i)$ can be defined for any of the geodesics in the family linking points on $(o_i)$ and $(s)$:

$$\Omega(SO_i) = \Omega(x'x_{(i)}) \equiv \Omega_i(v) = \frac{1}{2}(u_{Oi} - u_S)\int_{u_S}^{u_{Oi}} g_{\mu\nu}U^\mu U^\nu du, \tag{4}$$

taken along $\Gamma_{SO_i}(v)$ with $U^\mu = \dfrac{dx^\mu_{(i)}}{du}$, has a value independent of the particular affine parameter chosen. The holonomic metric $g = g_{\mu\nu}\,\vartheta^\mu \times \vartheta^\nu = g(e_\mu,\, e_\nu)\,\vartheta^\mu \times \vartheta^\nu$, of signature $+2$ (a chronometric interpretation), is defined in the Riemannian spacetime, with the components, $g_{\mu\nu} = g(e_\mu, e_\nu)$ in the dual holonomic basis $\{\vartheta^\mu \equiv dx^\mu\}$. We have taken $u_S = 0$ and $u_{Oi} \leq 1$ for given world function $\Omega_{(i)}(v)$, which becomes

$$\Omega_{(i)}(v) = \frac{1}{2}u_{Oi}\int_0^1 g_{\mu\nu}U^\mu U^\nu du. \tag{5}$$

By virtue of $\delta U^\mu/\delta u = 0$, we have $g_{\mu\nu}U^\mu U^\nu = const$ along $\Gamma_{SO_i}(v)$, therefore, (4) is reduced to

$$\Omega_{(i)}(v) = \frac{1}{2}(u_{O_{(i)}} - u_S)^2 g_{\mu\nu}U^\mu U^\nu, \tag{6}$$

with the last part evaluated anywhere on $\Gamma_{SO_i}(v)$. Taking $u_S = 0$ and $u_{Oi} = 1$, and applying conventional methods **?**, we then have

$$\Omega_{(i)}(v) = \frac{1}{2}g_{\mu\nu}U^\mu U^\nu = \frac{1}{2}\varepsilon L_i^2, \quad L_i = \int_S^{O_i} ds, \tag{7}$$

that is, to within the factor $\varepsilon = \pm 1$, the world-function is half the square of the measure, $L_i$, of geodesic joining $S$ and $O_i$. Synge has proved that the following relations hold in general:

$$\left.\frac{\partial\Omega_{(i)}(v)}{\partial x^\mu}\right|_S = -u_{Oi}\; g_{\mu\nu}\left.\frac{\partial x^\nu}{\partial u}\right|_S, \quad \left.\frac{\partial\Omega_{(i)}(v)}{\partial x^\mu}\right|_{O_i} = u_{Oi}\; g_{\mu\nu}\left.\frac{\partial x^\nu}{\partial u}\right|_{O_i}. \tag{8}$$

The right hand sides are invariant under transformation of the affine parameter. If the geodesic is not null, one has

$$\frac{\partial\Omega_{(i)}}{\partial x'^\mu} = -L_i\tau_{\mu(S)}, \quad \frac{\partial\Omega_{(i)}}{\partial x^\mu_{(i)}} = L_i\tau_{\mu(O_i)}, \tag{9}$$

where $\tau_{\mu(S)}$ and $\tau_{\mu(O_i)}$ are the unit tangent vectors to the geodesic at $S$ and $O_i$.

For null geodesics $\Gamma_{S_{(1)}O_{i(1)}}(v)$ and $\Gamma_{S_{(2)}O_{i(2)}}(v+\bigtriangleup v)$, in particular, the world functions $\Omega_{(i)}(v)$ does not change in the interval $v$ and $v+\bigtriangleup v$, therefore

$$\left.\frac{\partial\Omega_{(i)}(v)}{\partial x^\mu}\frac{dx^\mu}{dv}\right|_{O_i} + \left.\frac{\partial\Omega_{(i)}(v)}{\partial x^\mu}\frac{dx^\mu}{dv}\right|_S = 0, \tag{10}$$

which yields

$$p_{\mu(i)}V^\mu_{(i)} \bigtriangleup \tau_{O_i} - p_{\mu(S)}V^\mu_{(S)} \bigtriangleup \tau_S = 0, \tag{11}$$

where $V^\mu_{(i)} = dx^\mu/d\tau_{O_i}|_{O_{i(1)}}$ and $V^\mu_{(S)} = dx^\mu/d\tau_S|_{S_{(1)}}$ are the respective four-velocity vectors of observer $O_i$ and source $S$ (or world lines $(o_i)$ and $(s)$) at points $O_{i(1)}$ and $S_{(1)}$, $p^\mu_{(i)} = dx^\mu_{(i)}/du_i$ and $p^\mu_{(S)} = dx'^\mu/du_0$ are respective four-momenta of light ray (tangent to null geodesic) at the end points. Then, by virtue of (2), we obtain

$$1 + z_i = \frac{l_i}{l_0} = \frac{p_{\mu(S)}V^\mu_{(S)}}{p_{\mu(i)}V^\mu_{(i)}}. \tag{12}$$

At $i = n$, the equation (12) becomes a general coordinate independent definition of spectral shift (1).


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In studying further a set of null geodesics $\Gamma(v)$ with equations $x^\mu(u_i, v)$ (where $v = const$), we may deal with the deviation vector $\eta^\mu_{(i)}$ drawn from $O_{i(1)}S_{(1)}$ to $O_{i(2)}S_{(2)}$, and that we have along null geodesic

$$\eta_{\mu(i)}\frac{\partial x^\mu}{\partial u_i} = const. \tag{13}$$

The equation (13) yields

$$\eta_{\mu(i+1)}p^\mu_{(i+1)} = \eta_{\mu(i)}p^\mu_{(i)}. \tag{14}$$

Then

$$\begin{aligned} &\eta_{\mu(i+1)} = V^\mu_{(i+1)}l_{i+1}, \quad \eta_{\mu(i)} = V^\mu_{(i)}l_i, \\ &\eta_{\mu(i+1)}p^\mu_{(i+1)} = -E_{i+1}l_{i+1}, \quad \eta_{\mu(i)}p^\mu_{(i)} = -E_i l_i, \end{aligned} \tag{15}$$

where $E_i = -p_{\mu(i)}V^\mu_{(i)}$ is the energy of light ray relative to an observer $O_i$. Combining (2) and (15), we may write the ratio $(\lambda_{i+1}/\lambda_i)$ in terms of energy of photon and the world-function

$$\frac{\lambda_{i+1}}{\lambda_i} = \frac{l_{i+1}}{l_i} = \frac{E_i}{E_{i+1}} = \frac{p_{\mu(i)}V^\mu_{(i)}}{p_{\mu(i+1)}V^\mu_{(i+1)}} = \frac{\Omega_{\mu(i)}V^\mu_{(i)}}{\Omega_{\mu(i+1)}V^\mu_{(i+1)}}, \tag{16}$$

where $\Omega_{\mu(i)} = (u_{Oi} - u_S)U_\mu$. Therefore, the infinitesimal `relative´ spectral shift $\delta z_i$ between the observers $O_{i+1}$ and $O_i$ will be

$$\begin{aligned} \delta z_i = \frac{\delta\lambda_i}{\lambda_i} = \frac{\lambda_{i+1} - \lambda_i}{\lambda_i} = \frac{\delta l_i}{l_i} = \frac{l_{i+1} - l_i}{l_i} = \frac{p_{\mu(i)}V^\mu_{(i)}}{p_{\mu(i+1)}V^\mu_{(i+1)}} - 1 = \frac{\Omega_{\mu(i)}V^\mu_{(i)}}{\Omega_{\mu(i+1)}V^\mu_{(i+1)}} - 1 = \\ \frac{\tilde{\delta} z_i}{1 + z_i} \equiv \frac{z_{i+1} - z_i}{1 + z_i}. \end{aligned} \tag{17}$$

For definiteness, let consider case of $l_n > l_0$ (being red-shift, Fig. 1). In similar way, of course, we may treat a negative case of $l_n < l_0$ (being blue-shift), but it goes without saying that in this case a source is moving towards the observer. In first case, the observers at the points $O_{i(2)}$ $(i = 1, ..., n)$ should observe the monotonic increments of `relative´ spectral shifts $(\delta z_0, \delta z_1, \delta z_2, ..., \delta z_{n-1})$ when light ray passes, respectively, across the infinitesimal distances $(O_{1(2)}, S_{(2)}), (O_{2(2)}, O_{1(2)}), ..., (O_{n(2)}, O_{(n-1)(2)})$. Thus, the wavelength of light emitted at $S_{(2)}$ is stretched out observed at the points $O_{i(2)}$. While weak, such effects considered cumulatively over a great number of successive increments of `relative´ spectral shifts could become significant. The resulting spectral shift is the accumulation of a series of infinitesimal shifts as the light ray passes from luminous source to adjacent observers along the path of light ray. This interpretation holds rigorously even for large spectral shifts of order one or more. If this view would prove to be true, then it would lead to the chain rule for the wavelengths:

$$\frac{\lambda_{O_{(n2)}}}{\lambda_0} \equiv \frac{\lambda_n}{\lambda_0} = \frac{\lambda_n}{\lambda_{n-1}} \cdot \frac{\lambda_{n-1}}{\lambda_{n-2}} \cdots \frac{\lambda_2}{\lambda_1} \cdot \frac{\lambda_1}{\lambda_0} = \prod_{i=0}^{n-1}(1 + \delta z_i), \tag{18}$$

where $\lambda_0 \equiv \lambda_{S_{(2)}}$, such that

$$1 + z = \frac{\lambda_n}{\lambda_0} = \prod_{i=0}^{n-1}(1 + \delta z_i) = \prod_{i=0}^{n-1}\frac{p_{\mu(i)}V^\mu_{(i)}}{p_{\mu(i+1)}V^\mu_{(i+1)}} = -\prod_{i=0}^{n-1}\frac{\Omega_{\mu(i)}V^\mu_{(i)}}{\Omega_{\mu(i+1)}V^\mu_{(i+1)}}, \tag{19}$$

where $\Omega_{\mu(0)} = -(u_{O1} - u_S)U_{\mu(S)}$.

With no loss of generality, we may of course apply the equation (19) all the way to $n \to \infty$. Let us recast the increment of the proper space scale factor, $l_i = l(u_i)$, over the affine parameters $u_i$ $(i = 1, 2, ..., n)$, into the form $l_i = l_0 + i\varepsilon$, where $\varepsilon$ can be made arbitrarily small by increasing $n$. In the limit $n \to \infty$, all the respective adjacent observers are arbitrarily close to each other, so that $\delta z_i = \delta l_i/l_i \simeq \varepsilon/l_0 \to 0$. This allows us to write the following relation for the infinitesimal `relative´ redshifts:

$$\lim_{n\to\infty}(\delta z_{n-1} = \delta z_{n-2} = \cdots = \delta z_1 = \delta z_0 = \varepsilon/l_0) = \lim_{n\to\infty}\left(\delta z^{(a)}_{(n)} \equiv \frac{1}{n}\sum_{i=0}^{n-1}\delta z_i\right), \tag{20}$$


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provided, $\delta z^{(a)}_{(n)}$ is the average infinitesimal increment of spectral shift. There does not seem to be any reason to doubt a validity of (20). Certainly, the identification adopted here can be readily proved as follows. The relation (19) then becomes

$$\begin{aligned}1+z=\lim_{n\to\infty}\prod_{i=0}^{n-1}(1+\delta z_i)=\lim_{n\to\infty}\left(1+\delta z^{(a)}_{(n)}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\sum_{i=0}^{n}\frac{\delta l_i}{l_i}\right)^n=\\ \lim_{n\to\infty}\left(1+\frac{1}{n}\ln\frac{l_n}{l_0}\right)^n=\frac{l_n}{l_0}=\frac{p_{\mu(S)}V^{\mu}_{(S)}}{p_{\mu(O)}V^{\mu}_{(O)}},\end{aligned}\tag{21}$$

and hence

$$\begin{aligned}1+z=\frac{\triangle\,\tau_O}{\triangle\,\tau_S}=\frac{p_{\mu(S)}V^{\mu}_{(S)}}{p_{\mu(O)}V^{\mu}_{(O)}}=-\frac{\Omega_{\mu(S)}V^{\mu}_{(S)}}{\Omega_{\mu(O)}V^{\mu}_{(O)}}=1+\sum_{i=0}^{n-1}\tilde{\delta} z_i=\lim_{n\to\infty}\prod_{i=0}^{n-1}(1+\delta z_i)=\\ \lim_{n\to\infty}\left(1+\delta z^{(a)}_{(n)}\right)^n,\end{aligned}\tag{22}$$

where $\Omega_{\mu(O)}=(u_O-u_S)U_{\mu(O)}$ and $\Omega_{\mu(S)}=-(u_O-u_S)U_{\mu(S)}$. The first line of equation (22) is the overall spectral shift rule (1), which proves a validity of the relation (20).

It is worth emphasizing that the general equation (22) follows from a series of infinitesimal stretching of the proper space scale factor in Riemannian spacetime, whereas the path of a luminous source appears nowhere, thus this equation does not relate to the special choice of transport path. Therefore, to remove the ambiguity of parallel transport of four-velocities in curved spacetime, we advocate exclusively with this proposal. To obtain some feeling about this statement, below we give more detailed explanation. Imagine a family of adjacent observers $(O_{a_i}(u_i))$ situated at the points $a_i$ $(i=1,...,n)$ on the world lines $(o_i)$ at infinitesimal distances from the observers $(O_{i(2)})$, who measure the wavelength of radiation in relative motion caused by a series of infinitesimal stretching $(\delta l_0,...,\delta l_{n-1})$ of the proper space scale factor. Let $v_{O_{i(2)}O_{a_i}}(u_i)\equiv c\delta\beta_{i-1}=c(\beta_i-\beta_{i-1})$ be an infinitesimal increment of the velocity, $c\beta_i$, of observer $O_{i(2)}$ with respect to the velocity, $c\beta_{i-1}$, of a neighboring observer $O_{a_i}$ at the point $a_i$, due to infinitesimal stretching of the proper space scale factor $\delta l_{i-1}$. Since each line segment $l_{i-1}$ of proper space scale factor (at the affine parameter $u_{i-1}$) is identically mapped on the line segment $(l_i-\delta l_{i-1})$ (where $\delta l_i$ denotes infinitesimal segment $a_iO_{i(2)}$) of proper space scale factor (at the affine parameter $u_i=u_{i-1}+\delta u_{i-1}$), the relative velocity $v_{O_{i(2)}O_{a_i}}(u_i)$ of observer $(O_{i(2)}(u_i))$ to adjacent observer $(O_{a_i})(u_i)$ should be the same as it is relative to observer $(O_{(i-1)(2)}(u_{i-1}))$, that is $v_{O_{i(2)}O_{(i-1)}}(u_{i-1})\equiv v_{O_{i(2)}O_{a_i}}(u_i)$. Taking into account that the infinitesimal velocities of source $(S)$ relative to observers $(O_{i(2)})$ arise at a series of infinitesimal stretching of the proper space scale factor $\delta l_i$ $(i=0,1,2,...,n-1)$ as it is seen from the Fig. 1, we may fill out the whole pattern of monotonic increments of `relative´ spectral shifts $(\delta z_0,\delta z_1,\delta z_2,...,\delta z_{n-1})$ by, equivalently, replacing the respective pairs $(O_{1(2)},S_{(2)}),(O_{2(2)},O_{1(2)}),...,(O_{n(2)},O_{(n-1)(2)})$ with new ones $(O_{1(2)},O_{a_1}),(O_{2(2)},O_{a_2}),...,(O_{n(2)},O_{a_n})$, which attribute to the successive increments of relative velocities $v_{O_{1(2)}S}(u_1),...,v_{O_{n(2)}O_{(n-1)(2)}}(u_n)$ of the source $(S)$ away from an observer $(O_{n(2)})$ in the rest frame of $(O_{n(2)})$, viewed over all the values $(i=1,...,n)$. This framework provides the following definition.

*Definition:* The velocity $v_n\equiv v_{O_{n(2)}S_{(2)}}$ given in the limit $n\to\infty$ to be referred to as the relative velocity, $v_{r.v.}(z)$, of test particle (a source $(S)$) with respect to observer $(O_{n(2)})$ along the line of sight. That is

$$v_{r.v.}(z)=c\lim_{n\to\infty}\beta_n(z)\equiv\lim_{n\to\infty}v_{O_{n(2)}S_{(2)}}.\tag{23}$$

By virtue of the relation (20), at the limit $n\to\infty$, the relative infinitesimal velocities tends to zero, $v_{O_{i(2)}O_{a_i}}(u_i)=c\delta\beta_i=c\delta z_i(1-\beta_{i+1}\beta_i)\simeq(c\varepsilon/l_0)(1-\beta_{i+1}\beta_i)\to 0$, such that

$$\lim_{n\to\infty}\delta\beta_0=\lim_{n\to\infty}\delta\beta_1=\lim_{n\to\infty}\delta\beta_2=\cdots=\lim_{n\to\infty}\delta\beta_{n-1}=\lim_{n\to\infty}\left(\delta\beta^{(a)}_{(n)}\equiv\frac{1}{n}\sum_{i=0}^{n-1}\delta\beta_i\right)=\lim_{n\to\infty}\frac{1}{n}\beta_n.\tag{24}$$


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We are free to deal with any infinitesimal `relative´ spectral shift $\delta z_i$ for the pair $(O_{i(2)})$ and $(O_{a_i})$, in local tangent inertial rest frame of an observer $(O_{i(2)})$, where we may approximate away the curvature of space in the infinitesimally small neighborhood.

*Remark:* In this case, although the corresponding infinitesimal relative velocities arise as the first-order Doppler velocities, $\delta\beta_i^{(r)} = \delta z_i$, we cannot summing them over. Actually, the infinitesimal relative velocities arise in generic pseudo-Riemannian spacetime at a series of infinitesimal stretching of the proper space scale factor as alluded to above, so that the *SR law of composition of velocities cannot be implemented globally along non-null geodesic* because these velocities are velocities at the different events, which should be in a different physical frames, and cannot be added together.

To facilitate further calculations of the relative velocity in quest, we can address the pair of observers at points $O_{(n)2}$ and $a_n$. Suppose $V^\mu_{O_{n(2)}}$ and $V^\mu_{O_{a_n}}$ be the unit tangent four-velocity vectors of observers $(O_{n(2)})$ and $(O_{a_n})$ to the respective world-lines in a Riemannian spacetime, thus in their respective rest frame we have $V^0_{O_{n(2)}} = 1$ and $V^0_{O_{a_n}} = 1$, as the only nonzero components of velocity. The ray passes an observer $O_{a_n}(u_n)(\equiv O_{(n-1)(2)}(u_{n-1}))$ with the proper space scale factor $l_{n-1}$ who measures the wavelength to be $\lambda_{n-1}$. The ray passes next observer $O_{n(2)}(u_n)$ with the proper space scale factor $l_n = l_{n-1} + \delta l_{n-1}$. The ray's wavelength measured by observer $O_{n(2)}(u_n)$ is increased by $\delta\lambda_{n-1} = \lambda_n - \lambda_{n-1}$ leading to infinitesimal `relative´ spectral shift $\delta z_{n-1}$. For comparing the vectors $V^\mu_{O_{n(2)}}$ and $V^\mu_{O_{a_n}}$ at different events, it is necessary to seek a useful definition of the relative velocity by bringing both vectors to a common event by subjecting one of them to parallel transport. Since all the paths between infinitesimally separated spacetime points $O_{(n)2}$ and $a_n$ are coincident at $n \to \infty$, for comparing these velocities there is no need to worry about specific choice of the path of parallel transport of four-vector. Therefore, we are free to subject further the unit tangent four-velocity vector $V^\mu_{O_{a_n}}$ to parallel transport along the null geodesic $\Gamma_{a_n O_{n(2)}}$ to the point $O_{n(2)}$. A parallel transport yields at $O_{n(2)}$ the vector $\beta_{\mu(O_{n(2)})} = \mathrm{g}_{\mu\nu'}(O_{(n)2}, O_{a_n})V^{\nu'}_{O_{a_n}}$, where the two point tensor $\mathrm{g}_{\mu\nu'}(O_{(n)2}, O_{a_n})$ is the parallel propagator, which is determined by the points $O_{a_n}$ and $O_{(n)2}$. At $O_{a_n} \to O_{(n)2}$, we have the coincidence limit $[\mathrm{g}_{\mu\nu}](O_{n(2)}) = g_{\mu\nu}(O_{n(2)})$. As we have at point $O_{n(2)}$ two velocities $V^\mu_{O_{n(2)}}$ and $\beta^\mu_{(O_{n(2)})} = g^{\mu\nu}\beta_{\nu(O_{n(2)})}$, we may associate Doppler shift $\delta z_{n-1}$ to four-velocity $\beta^\mu_{(O_{n(2)})}$ of observer $O_{a_n}$ observed by an observer $O_{n(2)}$ with four-velocity $V^\mu_{O_{n(2)}}$ as measured by the latter. Then, following (Synge, 1960), the infinitesimal Doppler shift can be written:

$$\delta z_{n-1} = \frac{\delta\lambda_{n-1}}{\lambda_{n-1}} = \frac{p_{\mu(O_{n(2)})}\beta^\mu_{(O_{n(2)})}}{p_{\mu(O_{n(2)})}V^\mu_{(O_{n(2)})}} - 1 = \left[(1+\beta^2_{(O_{n(2)})})^{1/2} + \beta_{R(O_{n(2)})}\right] - 1, \tag{25}$$

where $c\beta^\mu_{(O_{n(2)})} = v^\mu_{(O_{n(2)})}$, $c\beta_{(O_{n(2)})} = v_{(O_{n(2)})}$, $c\beta_{R(O_{n(2)})} = v_{R(O_{n(2)})}$, and

$$\begin{aligned} &v^2_{(O_{n(2)})} = v_{(\alpha)(O_{n(2)})}v^{(\alpha)}_{(O_{n(2)})}, \quad v_{(\alpha)(O_{n(2)})} = v_{\mu(O_{n(2)})}\xi^\mu_{(\alpha)(O_{n(2)})}, \\ &v_{R(O_{n(2)})} = v_{\mu(O_{n(2)})}r^\mu_{(O_{n(2)})} = v_{(\alpha)(O_{n(2)})}v^{(\alpha)}_{(O_{n(2)})}. \end{aligned} \tag{26}$$

Reviewing notations the three-velocity of an observer $(O_{a_n})$ relative to observer at $(O_{n(2)})$ is $v_{(\alpha)(O_{n(2)})}$, the relative speed is $v_{(O_{n(2)})}$, and $v_{R(O_{n(2)})}$ is the speed of recession of $(O_{a_n})$. Whereas $\xi^\mu_{(\alpha)(O_{n(2)})}$ is the frame of reference on world-line $(o)$ with $\xi^\mu_{0(O_{n(2)})} = V^\mu_{(O_{n(2)})}$, the unit vector $r^\mu_{(O_{n(2)})}$ at $O_{n(2)}$ is orthogonal to world-line $(o)$ $(r_{\mu(O_{n(2)})}V^\mu_{(O_{n(2)})} = 0)$ and lying in the 2-element which contains the tangent at $O_{n(2)}$ to $(o)$ and $S_{(2)}O_{(2)}$.

In the local inertial rest frame $\xi^\mu_{(\alpha)(O_{n(2)})}$ of an observer $(O_{n(2)})$, the velocity vector $\beta^\mu_{O_{n(2)}}$ takes the form $(\gamma, \gamma\delta\beta_{(O_{n(2)})}, 00)$, where an observer $(O_{a_n})$ is moving away from the observer $(O_n(2))$ with the relative infinitesimal three-velocity $\delta\beta_{(O_{n(2)})}$ (in units of the speed of light) in a direction making an angel $\theta_{(O_{n(2)})}$ with the outward direction of line of sight $\Gamma_{O_{a_n}O_{n(2)}}$ from $O_{a_n}$ to $O_{(n)2}$, and $\gamma =$


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$(1-\delta\beta^2_{(O_{n(2)})})^{-1/2}$. Therefore, the equation (25) is reduced to

$$\delta z_{n-1} = \frac{1+\delta\beta_{(O_{n(2)})}\cos\theta_{(O_{n(2)})}}{\sqrt{1-\delta\beta^2_{(O_{n(2)})}}} - 1 = \beta_{R(O_{n(2)})} + \cdots \simeq \beta_{R(O_{n(2)})} = \frac{p_{(\alpha)(O_{n(2)})} v^{(\alpha)}_{(O_{n(2)})}}{E_{O_{n(2)}}} = \delta\beta_{(O_{n(2)})}\cos\theta_{(O_{n(2)})}. \tag{27}$$

Thus, at $n\to\infty$, the wavelength measured by the observer $O_{n(2)}$ is increased by the first-order Doppler shift caused unambiguously by the infinitesimal relative speed $\delta\beta^{(r)}_{n-1} \equiv \delta\beta_{(O_{n(2)})}\cos\theta_{(O_{n(2)})}$ along the line of sight with end-points $O_{a_n}$ and $O_{n(2)}$:

$$\delta z_{n-1} = \frac{\delta l_{n-1}}{l_{n-1}} = \delta\beta^{(r)}_{n-1}. \tag{28}$$

The SR law of composition of velocities along the line of sight can be implemented in the tangent inertial rest frame of an observer $O_{n(2)}$:

$$\delta\beta^{(r)}_{n-1} = \frac{\beta_n - \beta_{n-1}}{1-\beta_n\beta_{n-1}} \simeq \frac{\delta\beta_{n-1}}{1-\beta^2_{n-1}}, \tag{29}$$

where $v_{n-1} = c\beta_{n-1}$ and $v_n = c\beta_n$ are, respectively, the three-velocities of observers $O_{a_n}$ and $O_{n(2)}$ along the line of sight with end-points $O_{a_n}$ and $O_{n(2)}$. According to (24), at $n\to\infty$, a resulting infinitesimal increment $\delta z_{n-1}$ of spectral shift reads

$$\lim_{n\to\infty}\delta z_{n-1} = \lim_{n\to\infty}\frac{\delta\beta_{n-1}}{1-\beta^2_{n-1}} = \lim_{n\to\infty}\frac{\beta_n}{n(1-\beta^2_n)}, \tag{30}$$

A convenient form of relation (22), due to (20), can be written

$$1+z = \frac{p_{\mu S}V^\mu_S}{p_{\mu O}V^\mu_O} = -\frac{\Omega_{\mu(S)}V^\mu_{(S)}}{\Omega_{\mu(O)}V^\mu_{(O)}} = \lim_{n\to\infty}(1+\delta z_{n-1})^n. \tag{31}$$

This, combined with (30), gives a finite spectral shift

$$1+z = \frac{p_{\mu S}V^\mu_S}{p_{\mu O}V^\mu_O} = -\frac{\Omega_{\mu(S)}V^\mu_{(S)}}{\Omega_{\mu(O)}V^\mu_{(O)}} = \lim_{n\to\infty}\left[1+\frac{1}{n}\left(\frac{\beta_n}{1-\beta^2_n}\right)\right]^n = \exp\left(\frac{\beta_{r.v.}}{1-\beta^2_{r.v.}}\right). \tag{32}$$

where the subscript $()_{r.v.}$ designates relative velocity $\beta_{r.v.} \equiv \lim_{n\to\infty}\beta_n$. The equation (32) directly yields the relative velocity of test particle (a luminous source) as measured along the observer's line-of-sight in generic pseudo-Riemannian spacetime:

$$\beta_{r.v.} = \frac{\sqrt{1+4\left[\ln\left(p_{\mu S}V^\mu_S\right)-\ln\left(p_{\mu O}V^\mu_O\right)\right]^2}-1}{2\left[\ln\left(p_{\mu S}V^\mu_S\right)-\ln\left(p_{\mu O}V^\mu_O\right)\right]} = \frac{\sqrt{1+4\left[\ln\left(-\Omega_{\mu(S)}V^\mu_{(S)}\right)-\ln\left(\Omega_{\mu(O)}V^\mu_{(O)}\right)\right]^2}-1}{2\left[\ln\left(-\Omega_{\mu(S)}V^\mu_{(S)}\right)-\ln\left(\Omega_{\mu(O)}V^\mu_{(O)}\right)\right]}. \tag{33}$$

The relative velocity of test particle is plotted on the Fig. 2 for spectral shifts $-1\le z\le 4$.

### 2.1. A global Doppler velocity along the null geodesic

A final point should be noted. Suppose the velocities of observers say $O_{i(2)}$ $(i=1,...,n-1)$, being in free fall, populated along the null geodesic $\Gamma_{S_{(2)}O_{(2)}}(v+\triangle v)$ of light ray (Fig. 1), vary smoothly along the line of sight with the infinitesimal increment of relative velocity $\delta\beta^{(r)}_i$. The $(i)$-th observer situated at the point $i(2)$ of intersection of the ray's trajectory $\Gamma_{S_{(2)}O_{(2)}}(v+\triangle v)$ with the world line $(o_i)$ at affine parameter $u_i$, and measures the frequency of light ray as it goes by. According to the equivalence principle, we may approximate away the curvature of space in the infinitesimally small neighborhood


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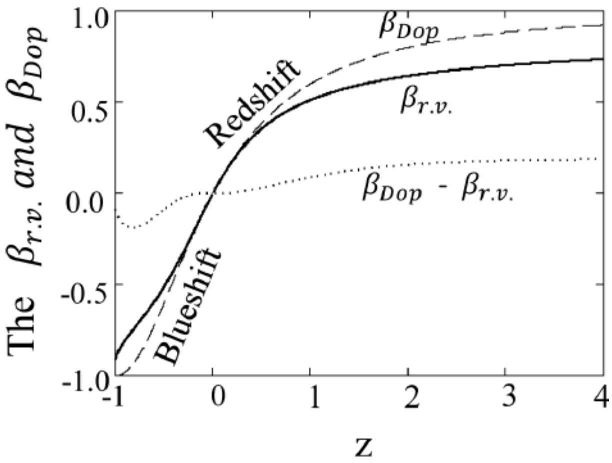


Figure 2. The relative velocity along the line of sight ($\beta_{r.v.}$) vs. $z$. The global Doppler velocity ($\beta_{Dop}$), and their difference (in units of the speed of light) are also presented.

of two adjacent observers. If we approximate an infinitesimally small neighborhood of a curved space as flat, the resulting errors are of order $(\delta l_i/l_n)^2$ in the metric. If we regard such errors as negligible, then we can legitimately approximate spacetime as flat. The infinitesimal increment of spectral shift $\delta z_i$ is not approximated away in this limit because it is in that neighborhood of leading order $(\delta l_i/l_i)$. That is, approximating away the curvature of space in the infinitesimally small neighborhood does not mean approximating away the infinitesimal increment $\delta z_i$. Imagine a thin world tube around the null geodesic $\Gamma_{S_{(2)}O_{(2)}}(v+\bigtriangleup v)$ within which the space is flat to arbitrary precision. Each observer has a local reference frame in which SR can be taken to apply, and the observers are close enough together that each one $O_{i(2)}$ lies within the local frame of his neighbor $O_{(i+1)(2)}$. This implies the vacuum value of a velocity of light to be universal maximum attainable velocity of a material body found in this space. Such statement is true for any thin neighborhood around a null geodesic. Only in this particular case, the relative velocity of observers can be calculated by the SR law of composition of velocities globally along the path of light ray. We may apply this law to relate the velocity $\beta_i$ to the velocity $\beta_{i+1}$, measured in the $(i+1)$-th adjacent observer's rest frame. The end points of infinitesimal distance between the adjacent observers $O_{i(2)}$ and $O_{(i+1)(2)}$ will respectively be the points of intersection of the ray's trajectory with the world lines $o_i(u_i)$ and $(o_{i+1})(u_{i+1})$. This causes a series of infinitesimal increment of the proper space scale factor from $l_i =\bigtriangleup \tau_i$ to $l_{i+1} =\bigtriangleup \tau_{i+1}$, which in turn causes a series of infinitesimal increment of spectral shift $\delta z_i = \delta\lambda_i/\lambda_i = \delta l_i/l_i$. Within each local inertial frame, there are no gravitational effects, and hence the infinitesimal spectral shift from each observer to the next is a Doppler shift. Thus, at the limit $n\to\infty$, a resulting infinitesimal frequency shift $\delta z_i$, can be unambiguously equated to infinitesimal increment of a fractional SR Doppler shift $\delta\bar{z}_i$ from observer $O_{i(2)}$ to the next $O_{(i+1)(2)}$ caused by infinitesimal relative velocity $\delta\bar{\beta}^r_i$:

$$\left(\delta z_i = \frac{\delta l_i}{l_i}\right)_{n\to\infty} = \left(\delta\bar{z}_i = \delta\bar{\beta}^r_i = \frac{\bar{\beta}_{i+1}-\bar{\beta}_i}{1-\bar{\beta}_{i+1}\bar{\beta}_i} \simeq \frac{\delta\bar{\beta}_i}{1-\bar{\beta}^2_i}\right)_{n\to\infty}, \tag{34}$$

where by $\bar{()}$ we denote the null-geodesic value, as different choices of geodesics yield different results for the motion of distant test particles relative to a particular observer. The relation (34), incorporated with the identity (20), yield

$$\left(\delta z_{n-1} = \delta\beta^{(r)}_{n-1}\right)_{n\to\infty} = \left(\frac{\delta\beta_{n-1}}{1-\beta^2_n}\right)_{n\to\infty} = \left(\delta\bar{z}^{(a)}_{(n)} = \delta\bar{\beta}^{r(a)}_{(n)} \equiv \frac{1}{n}\sum_{i=0}^{n-1}\frac{\delta\bar{\beta}_i}{1-\bar{\beta}^2_i}\right)_{n\to\infty}, \tag{35}$$

which, by virtue of (24), for sufficiently large but finite $n$ gives

$$\frac{\beta_n}{1-\beta^2_n} = \sum_{i=0}^{n-1}\frac{\delta\bar{\beta}_i}{1-\bar{\beta}^2_i} = \int_0^{\bar{\beta}_n}\frac{d\bar{\beta}}{1-\bar{\beta}^2}, \tag{36}$$


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or

$$\bar{\beta}_n = \frac{e^{\varrho_n} - 1}{e^{\varrho_n} + 1}, \quad \varrho_n \equiv \frac{2\beta_n}{1-\beta_n^2}. \tag{37}$$

Hence the general solution (39), by means of relation (36), is reduced to a global Doppler shift along the null geodesic:

$$1 + z = \sqrt{\frac{1+\bar{\beta}_{r.v.}}{1-\bar{\beta}_{r.v.}}} = \frac{p_{\mu(O2)}V^{\mu}_{(S2)}}{p_{\mu(O2)}V^{\mu}_{(O2)}}, \tag{38}$$

where $\bar{\beta}_{r.v.} = \lim_{n\to\infty} \bar{\beta}_n$, $V^{\mu}_{(S2)}$ and $V^{\mu}_{(O2)}$ are the four-velocity vectors, respectively, of the source $S_{(2)}$ and observer $O_{(2)}$, $p_{\mu(S2)}$ and $p_{\mu(O2)}$ are the tangent vectors to the typical null geodesics $\Gamma_{S_{(2)}O_{(2)}}(v)$ at their respective end points. This procedure, in fact, is equivalent to performing parallel transport of the source four-velocity in a general Riemannian spacetime along the null geodesic to the observer. Note that any null geodesic from a set of null geodesics mapped $(s)$ on $(o)$ can be treated in the similar way.

In Minkowski space a parallel transport of vectors is trivial and mostly not mentioned at all. This allows us to apply globally the SR law of composition of velocities to relate the velocities $\bar{\beta}_i$ to the $\bar{\beta}_{i+1}$ of adjacent observers along the path of light ray, measured in the $(i+1)$-th adjacent observer's frame. Then, according to (34)-(38), a global Doppler shift of light ray emitted by luminous source as it appears to observer at rest in flat Minkowski space can be derived by summing up the infinitesimal Doppler shifts caused by infinitesimal relative velocities of adjacent observers.

## 3. The several special cases

The equation (32) directly yields the relative velocity of test particle (a luminous source) as measured along the observer's line-of-sight in generic pseudo-Riemannian spacetime:

$$\beta_{r.v.} = \frac{\sqrt{1+4\left[\ln\left(p_{\mu S}V^{\mu}_{S}\right) - \ln\left(p_{\mu O}V^{\mu}_{O}\right)\right]^2} - 1}{2\left[\ln\left(p_{\mu S}V^{\mu}_{S}\right) - \ln\left(p_{\mu O}V^{\mu}_{O}\right)\right]} = \frac{\sqrt{1+4\left[\ln\left(-\Omega_{\mu(S)}V^{\mu}_{(S)}\right) - \ln\left(\Omega_{\mu(O)}V^{\mu}_{(O)}\right)\right]^2} - 1}{2\left[\ln\left(-\Omega_{\mu(S)}V^{\mu}_{(S)}\right) - \ln\left(\Omega_{\mu(O)}V^{\mu}_{(O)}\right)\right]}. \tag{39}$$

In this section we discuss the implications of the general solution (39) for several instructive cases of potential interest, that is the Minkowski metric, the test particle and observer at rest in an arbitrary stationary metric, the uniform gravitational field, the rotating reference frame, the Schwarzschild metric, the Kerr-type metrics, and the spatially homogeneous and isotropic Robertson-Walker spacetime of standard cosmological model. We use the term `Doppler shift´ in a generalized sense, to refer to any effect that causes a photon's frequency at detection to differ from that which it had. Unless otherwise stated we take, for convenience natural, units, $h = c = G = 1$.

Suppose the metric $g_{\mu\nu}$ is stationary, i.e. there exists a coordinate system such that the metric tensor is independent of the time coordinate $x^0$. Let also $\mathbf{U} = d\mathbf{x}/d\tau$ be the four-velocity of an observer carrying the clock, which in general is written

$$\mathbf{U} = (-g_{00} - 2g_{i0}v^i - g_{ij}v^iv^j)^{1/2}(1, \vec{v}), \tag{40}$$

where $d\tau = (-g_{\mu\nu}dx^{\mu}dx^{\nu})^{1/2}$ is the proper time, measured on a clock moving with three-velocity $\vec{v} = d\vec{x}/dx^0$ in an arbitrary coordinate system:

$$d\tau = (-g_{00} - 2g_{i0}v^i - g_{ij}v^iv^j)^{1/2}, \tag{41}$$

provided, $dx^0$ is the coordinate time interval. The energy $\hat{E}$ of test particle relative to an observer, with four-momentum $\mathbf{P} = E(1, \vec{w})$, can be written

$$\hat{E} = -\mathbf{U}\cdot\mathbf{P} = \frac{(g_{00} + g_{i0}v^i + g_{i0}w^i + g_{ij}v^iw^j)P^0}{(-g_{00} - 2g_{i0}v^i - g_{ij}v^iv^j)^{1/2}}, \tag{42}$$


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where $E$ and $\vec{w}$ are the energy and spatial coordinate velocity of an observer. If test particle is freely moving in a time-independent metric, then from the equations of motion it follows that the covariant momentum $P_0$, conjugate to the time coordinate, is a constant of motion for the particle. The equation (42) then becomes

$$\hat{E} = \frac{(g_{00} + g_{i0}v^i + g_{i0}w^i + g_{ij}v^i w^j)P_0}{(-g_{00} - 2g_{i0}v^i - g_{ij}v^i v^j)^{1/2}(g_{00} + g_{i0}w^i)}. \tag{43}$$

By virtue of (43) and that $P_0$ is a constant of motion, the spectral shift rule (1) gives the frequency shift $\omega_O D_O = \omega_S D_S$ (Grøn, 1980), where $D$ is a general Doppler shift factor:

$$D = \frac{(-g_{00} - 2g_{i0}v^i - g_{ij}v^i v^j)^{1/2}(g_{00} + g_{i0}w^i)}{(g_{00} + g_{i0}v^i + g_{i0}w^i + g_{ij}v^i w^j)}. \tag{44}$$

Therefore, we may recast the general solution (39) into the form

$$\beta_{r.v.} = \frac{\sqrt{1 + 4\left(\ln D_O - \ln D_S\right)^2} - 1}{2\left(\ln D_O - \ln D_S\right)}, \tag{45}$$

### 3.1. The Minkowski metric

In case at hand, there is no deflection of the light, and the magnitude of the velocity of light is constant, so $\vec{w}_O = \vec{w}_S = \vec{n}$, where $\vec{n}$ is a unit vector in the direction of propagation of the light. Then $D = \gamma(1 - \vec{v} \cdot \vec{n})$, where $\gamma = (1 - v^2)^{-1/2}$ (see Grøn, 1980). Hence the general solution (45) reduced to

$$\beta_{r.v.} = \frac{\sqrt{1 + 4\left[\ln(\gamma_S(1 - \vec{v}_S \cdot \vec{n}) - \ln(\gamma_O(1 - \vec{v}_O \cdot \vec{n})\right]^2} - 1}{2\left[\ln(\gamma_S(1 - \vec{v}_S \cdot \vec{n}) - \ln(\gamma_O(1 - \vec{v}_O \cdot \vec{n})\right]}. \tag{46}$$

In the rest frame of observer $\vec{v}_O = 0$, the equation gives

$$\beta_{r.v.} = \frac{\sqrt{1 + 4\left[\ln(1 - \vec{v}_S \cdot \vec{n}) - \ln(1 - v_S^2)^{1/2}\right]^2} - 1}{2\left[\ln(1 - \vec{v}_S \cdot \vec{n}) - \ln(1 - v_S^2)^{1/2}\right]}. \tag{47}$$

The condition $\vec{v}_S \cdot \vec{n} = 0$ reveals the transverse Doppler effect- the SR time dilation. The case $\vec{v}_S \cdot \vec{n} = v_S$, describes the longitudinal Doppler shift (38).

### 3.2. The source and observer at rest in an arbitrary stationary metric

The spectral shift at $v_S = v_O = 0$ is known as the gravitational Doppler effect, such that the rate of time at the position of the absorber is different from that at the position of the emitter. Thereby (45) reduced to

$$\beta_{r.v.} = \frac{\sqrt{1 + \left(\ln g_{00}^O - \ln g_{00}^S\right)^2} - 1}{\ln g_{00}^O - \ln g_{00}^S}. \tag{48}$$

### 3.3. The uniform gravitational field

The case of uniform gravitational field can be identified with a rigidly accelerated reference frame in flat spacetime (Greenberger & Overhauser, 1979, Grøn, 1979). The metric in this frame with constant acceleration $g$ along the $z$ axis, as measured in its instantaneous rest inertial frames, is (Möller, 1952)

$$ds^2 = -(1 + gz)^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{49}$$

Thereby the time passes more slowly closer to the source in the gravitational field.


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The equations (45), for metric (49), yields

$$\beta_{r.v.} = \frac{\sqrt{1+4\ln^2\left\{\frac{[(1+gz_O)^2-v_O^2]^{1/2}(1+gz_O)^2[(1+gz_S)^2-v_S\cdot w_S]}{[(1+gz_S)^2-v_S^2]^{1/2}(1+gz_S)^2[(1+gz_O)^2-v_O\cdot w_O]}\right\}}-1}{2\ln\left\{\frac{[(1+gz_O)^2-v_O^2]^{1/2}(1+gz_O)^2[(1+gz_S)^2-v_S\cdot w_S]}{[(1+gz_S)^2-v_S^2]^{1/2}(1+gz_S)^2[(1+gz_O)^2-v_O\cdot w_O]}\right\}}. \tag{50}$$

Then, for both the test particle and observer at rest, the equation (50) gives

$$\beta_{r.v.} = \frac{\sqrt{1+4\left[\ln(1+gz_O)-\ln(1+gz_S)\right]^2}-1}{2\left[\ln(1+gz_O)-\ln(1+gz_S)\right]}. \tag{51}$$

### 3.4. The rotating reference frame

Consider a reference frame $\Sigma_r$, with cylindrical coordinates $(t, r, \theta, z)$ rotates with constant angular velocity $\omega$. The cylindrical coordinates are defined by the transformation from coordinates $(T, R, \Theta, Z)$ of inertial frame $\Sigma_0$, in which the axis of $\Sigma_r$, is permanently at rest: $T = t, \quad R = r, \quad \Theta = \theta + \omega t, \quad Z = z$. This gives for the line element in $\Sigma_r$

$$ds^2 = -(1-r^2\omega^2)dt^2 + dr^2 + r^2 d\theta^2 + dz^2 + 2\omega r^2 d\theta dt. \tag{52}$$

If the test particle and observer are both at rest in $\Sigma_r$, at distances $r_S$ and $r_O$. from the axis, respectively, equation (45) gives

$$\beta_{r.v.} = \frac{\sqrt{1+\left[\ln(1-r_O^2\omega^2)-\ln(1-r_S^2\omega^2)\right]^2}-1}{\ln(1-r_O^2\omega^2)-\ln(1-r_S^2\omega^2)}. \tag{53}$$

Note that:

(i) As $v^i$ and $w^i$ are components of coordinate velocities, so they need not have the dimension of (length/time). For example $v^i$ is an angular velocity with dimension time$^{-1}$. When given in an inertial frame $\Sigma_0$, they may be found in the arbitrary frame $\Sigma$ by using the coordinate transformation from $\Sigma_0$ to $\Sigma$.

(ii) The $\vec{w}$ is a spatial tangent vector along the path of a photon. Although $\vec{w}$ is always a unit vector in an inertial frame, this is not generally the case.

(iii) As shown by (Möller, 1952) it is not possible by a simple change of rate of coordinate clocks to introduce a time-orthogonal system of coordinates in the rotating reference frame. The non-vanishing $g_{02}$ component represents a genuine physical effect, and may be regarded as a component of a gravitational spatial vector potential, giving rise to a Coriolis acceleration for a moving particle in $\Sigma_r$.

### 3.5. The Schwarzschild metric

The line element in the usual Schwarzschild coordinates has the form

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{54}$$

where $B(r) = 1 - 2m/r$, $A(r) = 1/B(r)$. In this case the expression (44) becomes

$$\begin{aligned} \frac{D_O}{D_S} = & \frac{\left[B - A(v^r)^2 - r^2(v^\theta)^2 - r^2\sin^2\theta(v^\phi)^2\right]_O^{1/2} B_O}{\left[B - A(v^r)^2 - r^2(v^\theta)^2 - r^2\sin^2\theta(v^\phi)^2\right]_S^{1/2} B_S}\times \\ & \frac{\left(-B + Av^r w^r + r^2 v^\theta w^\theta + r^2\sin^2\theta v^\phi w^\phi\right)_S}{\left(-B + Av^r w^r + r^2 v^\theta w^\theta + r^2\sin^2\theta v^\phi w^\phi\right)_O}. \end{aligned} \tag{55}$$


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In case $v_S^\phi = w_O^\phi = w_S^\phi = w_O^\phi = 0$, the equation (55) can be written in the form (see Jaffe & Vessot, 1976)

$$\beta_{r.v.} = \frac{\sqrt{1+4\left[\ln\left(\frac{\left[B-A(v^r)^2-r^2(v^\theta)^2\right]_O^{1/2}\left[1-(A/B)^{1/2}(1-Bl^2/r^2)^{1/2}v^r-lv^\theta\right]_S}{\left[B-A(v^r)^2-r^2(v^\theta)^2-\right]_S^{1/2}\left[1-(A/B)^{1/2}(1-Bl^2/r^2)^{1/2}v^r-lv^\theta\right]_O}\right)\right]^2}-1}{2\left[\ln\left(\frac{\left[B-A(v^r)^2-r^2(v^\theta)^2\right]_O^{1/2}\left[1-(A/B)^{1/2}(1-Bl^2/r^2)^{1/2}v^r-lv^\theta\right]_S}{\left[B-A(v^r)^2-r^2(v^\theta)^2-\right]_S^{1/2}\left[1-(A/B)^{1/2}(1-Bl^2/r^2)^{1/2}v^r-lv^\theta\right]_O}\right)\right]}, \tag{56}$$

where $l$ is the impact parameter of the photon, as measured at infinity (Misner et al., 1973).

### 3.6. The Kerr-type metrics

Various frequency-shift effects have to be taken into account in a wide array of astrophysical contexts. The Kerr–Doppler effect has been studied in (Asaoka, 1989, Cisneros et al., 2015, Cunningham, 1998, Fanton et al., 1997, Li et al., 2005, Schönenbach, 2014). A formula for Kerr–Doppler effect for Kerr-type, (i.e. stationaryand axisymmetric) space–times, e.g. the coarse-grained, diffuse internal space–time of a spiral galaxy, is derived by (Cisneros et al., 2015), for the combined motional and gravitational Doppler effect in general stationary axisymmetric metrics for a photon emitted parallel or antiparallel to the assumed circular orbital motion of its source. In obtaining the formula, the authors utilized two seemingly different approaches, an eikonal approximation solution to a scalar wave equation in the Kerr-type metric, and a KV representation of both the source circular motion and the photon motion. Killing vector approach derivation will take advantage of the conserved quantities of time-like and null geodesic motion which result from the one parameter families of symmetries of the space–time, described by the KV fields. The two approaches produced the same formula, because despite apparent dissimilarities the underlying physics is the same. For example, the local propagation, or wave 3-vector used in the eikonal approach is (proportional to) the local photon 3-momentum used in the KV approach. While the KV approach is limited to the particular highly symmetric application that we treated, that of a photon emitted tangentially to the circular orbit of a source, it allows analysis in a more modern relativistic context, in which the relationship between the orbital velocity, $\Omega$, and the radius is determined by the conserved quantities of the Lagrangian. On the other hand, the eikonal method should be applicable for a local photon propagation 3-vector in any direction relative to the source motion, for the special case of the exterior Kerr metric. The wave equation would then yield a different expression for the local effective refractive index than it is obtained for tangential emission Note that the eikonal method should be applicable for a local photon propagation 3-vector in any direction relative to the source motion, for the special case of the exterior Kerr metric. This analysis extends to arbitrary Kerr-type (i.e. stationary and axisymmetric) space–times, e.g. the coarse-grained, diffuse internal space–time of a spiral galaxy. The formula yields expected results in the limits of a moving or stationary source in the exterior Kerr and Schwarzschild metrics and is useful for broad range astrophysical analyses.

The geometries under consideration comprise all stationary axisymmetric metrics of the Kerr type, i.e. all metrics independent of time $t$ and azimuthal angle $\varphi$ in polar (Boyer–Lindquist) coordinates $(t, r, \theta, \varphi)$, with $g_{0\varphi} = g_{\varphi 0}$ the only non-vanishing off-diagonal elements. In such metrics, the source was restricted to be moving in the $\varphi$-direction (which is the case for emitters in circular orbits in the equatorial plane, and also for emitters at the apsides of other orbits in that plane), and emitting in (or against) that same direction. For stable, circular, equatorial orbits that most closely approximate those in spiral galaxies, the relevant KV are the time-like $\xi = (1, 0, 0, 0)$ and the axial $\eta = (0, 0, 0, 1)$, for Kerr type metrics in Boyer–Lindquist-type coordinates. In this setting, the Doppler shift observed by a receiver in asymptotic flat space can be given without recourse to a perturbative expansion. Based on this analysis, the general formula (45), for a relative velocity of test particle moving directly


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towards or directly away from the asymptotic observer, can be recast into the form

$$\beta_{r.v.} = \frac{\sqrt{1+4\ln^2\left(\dfrac{g_{00}\sqrt{-g_{00}-2\Omega g_{0\varphi}-\Omega^2 g_{\varphi\varphi}}}{g_{00}+\Omega\left(g_{0\varphi}+\sqrt{g_{t\varphi}^2-g_{\varphi\varphi}g_{00}}\right)}\right)}-1}{2\ln\left(\dfrac{g_{00}\sqrt{-g_{00}-2\Omega g_{0\varphi}-\Omega^2 g_{\varphi\varphi}}}{g_{00}+\Omega\left(g_{0\varphi}+\sqrt{g_{t\varphi}^2-g_{\varphi\varphi}g_{00}}\right)}\right)}, \tag{57}$$

where $\Omega$ is the angular velocity $\Omega = d\varphi/dt = u^\varphi/u^t$, which is a constant for a circular orbit. In the case of Kerr black hole (Chandrasekhar, 1983, O'Neill, 1995), the metric components are written

$$\begin{aligned} &g_{00} = -(1-2Mr/\Sigma), \quad g_{\varphi\varphi} = \left[(r^2+a^2)^2 - \right. \\ &\left. a^2\Delta\sin^2\theta\right]\frac{\sin^2\theta}{\Sigma}, \quad g_{\theta\theta} = \Sigma, \quad g_{rr} = \Sigma/\Delta, \\ &g_{0\varphi} = g_{\varphi 0} = -2Mar\frac{\sin^2\theta}{\Sigma}, \end{aligned} \tag{58}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 + a^2 - 2Mr, \tag{59}$$

and for an emitter in the equatorial plane, $\theta = \pi/2$, Doppler formula derived in (Cisneros et al., 2015) coincides with the expression given in (Asaoka, 1989, Cunningham, 1998, Fanton et al., 1997, Li et al., 2005). To obtain the Schwarzschild metric limit we set $a = 0$ with $M = M_S/c^2$ for a source of mass $M_S$, then equation (57) yields

$$\beta_{r.v.} = \frac{\sqrt{1+4\ln^2\left[(1-2M/r)\left(\dfrac{(1-2M/r)^{1/2}+\Omega r/c}{(1-2M/r)^{1/2}-\Omega r/c}\right)\right]^{-1/2}}-1}{2\ln\left[(1-2M/r)\left(\dfrac{(1-2M/r)^{1/2}+\Omega r/c}{(1-2M/r)^{1/2}-\Omega r/c}\right)\right]^{-1/2}}, \tag{60}$$

where the velocity of light, $c$, is recovered. As expected, the equation (60) leads to the correct limits: the familiar gravitational redshift for non-moving optical sources ($\Omega = 0$), and the usual longitudinal Lorentz–Doppler ratio for $M = 0$ but $\Omega r/c = v/c$, where the relative source–observer velocity $v \approx \Omega r$ can be positive or negative. For a source in circular orbit, $M/r \approx v^2/c^2$, the equation (60) yields the usual Lorentz–Doppler formula to first order in $v/c$.

### 3.7. The Robertson-Walker spacetime

In the framework of standard cosmological model, one assumes that the universe is populated with comoving observers. In the homogeneous, isotropic universe comoving observers are in freefall, and obey Wayl's postulate: their all worldlines form a 3-bundle of non-intersecting geodesics orthogonal to a series of spacelike hypersurfaces, called comoving hypersurfaces. In case of expansion, all worldlines are intersecting only at one singular point. The clocks of comoving observers, therefore, can be synchronized once and for all. Let the proper time, $t$, of comoving observers be the temporal measure. Suppose $R(t)$ is the scale factor in expanding homogenous and isotropic universe. One considers in, so-called, cosmological rest frame a light that travels from a galaxy to a distant observer, both of whom are at rest in comoving coordinates. As the universe expands, the wavelengths of light rays are stretched out in proportion to the distance $L(t)$ between co-moving points ($t > t_1$), which in turn increase proportionally to $R(t)$ (Harrison, 1993, 1995):

$$\frac{\lambda(t)}{\lambda(t_1)} = \frac{dt}{dt_1} = \frac{R(t)}{R(t_1)} = \frac{L(t)}{L(t_1)}. \tag{61}$$

Reviewing notations in this `cosmic wavelength stretching´ relation with a fixed comoving coordinate $\chi$ ($d\chi = 0$), $L_1 \equiv L(t_1) = cR_1\chi$ is the proper distance to the source at the time when it emits light,


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$L(t) = cR(t)\chi$ is the same distance to the same source at light reception. Integration of differential equation (61) gives $\chi = \tau_1 - \tau$, where the interval of conformal time $d\tau \equiv dt/R(t) = \chi dt/L(t)$ is constant, i.e. at emission $d\tau_1$ equals $d\tau$ at reception.

In what follows, the mathematical structure has much in common with those constructions used for deriving of (17)-(39). After making due allowances for (61), particularly, the infinitesimal `relative´ increment $\delta z_j$ $(j = 1, ..., n-1)$ of redshift reads

$$\delta z_j = \frac{\delta\lambda_j}{\lambda_j} = \frac{\lambda_{j+1} - \lambda_j}{\lambda_j} = \frac{\delta L_j}{L_j} = \frac{L_{j+1} - L_j}{L_j} = \frac{\tilde{\delta} z_j}{1+z_j} \equiv \frac{z_{j+1} - z_j}{1+z_j}, \quad 1 + z_j = \frac{\lambda_j}{\lambda_1}, \tag{62}$$

where, the role of proper space scale factor $l_i$ is now destined to the scale factor $R(t_i) \propto L(t_i)$. As a corollary, the general relation (39) straightforwardly yields the particular solution for the case of expanding RW spacetime of standard cosmological model, i.e. the so-called *kinetic* recession velocity $v_{rec}$ of luminous source, which can be written in terms of scale factor $R(t)$:

$$v_{rec}(R) = \frac{\sqrt{1 + 4[\ln R(t) - \ln R_1]^2} - 1}{2[\ln R(t) - \ln R_1]}, \tag{63}$$

agreed with the result obtained by quite different study of, so-called, `lookforward´ history of expanding universe (Ter-Kazarian, 2021a, 2022). This interpretation so achieved has physical significance as it agrees with a view that the light waves will be stretched by travelling through the expanding universe, and in the same time the *kinetic* recession velocity of a distant astronomical object is always subluminal even for large redshifts of order one or more. It, therefore, does not violate the fundamental physical principle of *causality*. Moreover, the general solution is reduced to global Doppler shift (38) along the null geodesic, studied by Synge (Synge, 1960) (see also (Bunn & Hogg, 2009, Narlikar, 1994).

Once we are equipped with the general solution (63), we may define the most important parameter $\zeta(z, \dot{L})$ of practical measure of sweeping up of test particle with redshift $z$, in expanding universe:

$$\zeta(R, \dot{L}) \equiv \frac{v_{rec}(R)}{\dot{L}} = \frac{\sqrt{1 + 4[\ln R(t) - \ln R_1]^2} - 1}{2\dot{L}[\ln R(t) - \ln R_1]}, \tag{64}$$

where $\dot{L}(t, z)$ is the `proper´ recession velocity. The general solution (63) and parameter $\zeta(z, \dot{L})$ are plotted on the Fig. 3 for the distances at which the Hubble empirical linear `redshift-distance´ law $(cz = HL)$ is valid. (*Top panel(a):* for redshifts $0 \leq z \leq 25$; and *Bottom panel(b):* for redshifts $0 \leq z \leq 800$), where the global Doppler velocity, and their difference are also presented to guide the eye. As it is seen from the Fig. 3, the $\zeta(z) = 0.034$ at $z = 25$, and it declines in magnitude to $\zeta(z) = 1.16 \times 10^{-3}$ at $z = 800$. In particular, this offers a clear way out of ambiguity of the meaning of `expansion of flat 3D space´. Indeed, since the concept of 3D `space´ is not a well-defined (invariant) concept in general relativity, this arises the question whether the cosmological redshift in expanding universe can be distinguished from a purely kinematic Doppler effect resulting from the motion of test particles (galaxies) in stationary spacetime. The measure $\zeta(z)$, which declines in magnitude, uniquely distinguishes between real cosmological expansion and the hypothetical motion of galaxies in stationary spacetime. It proves that cosmological expansion of a flat 3D–space is fundamentally different from a kinematics of galaxies moving in a non-expanding flat 3D-space, in agreement with (Abramowicz et al., 2007). According to them, the rather wide-spread belief that cosmological expansion of a flat 3D–space (with spatial curvature k = 0) cannot be observationally distinguished from a kinematics of galaxies moving in a flat and non-expanding space is erroneous. The authors show that the expanding universe is necessarily a curved spacetime, so that the interpretation of the observed cosmological redshift as being due to the expansion of the cosmological 3D–space is observationally verifiable. Thus it is impossible to mimic the true cosmological redshift by a Doppler effect caused by motion of galaxies in a non-expanding 3D-space, flat or curved.

## 4. Concluding remarks

Let us briefly summarize the main results of this work. This report is about the much-discussed in literature question of the velocity of test particle relative to the observer in curved spacetime. We


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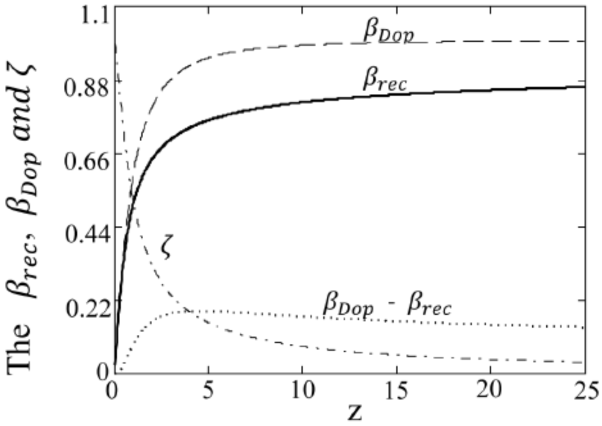


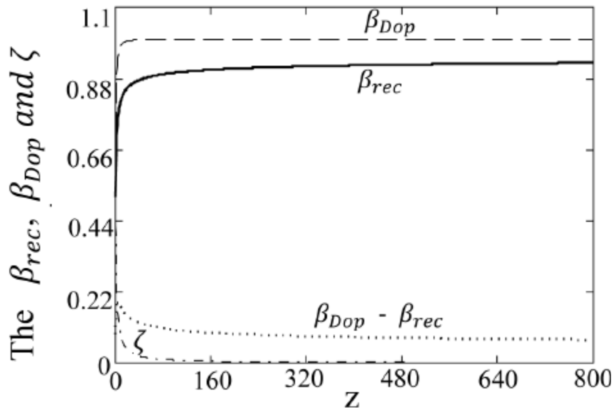


Figure 3. The parameter $\zeta$, the *kinetic* recession velocity ($\beta_{rec}$), the Doppler velocity ($\beta_{Dop}$), and their difference vs. $z$.
*Top panel:* $0 \leq z \leq 25$; *Bottom panel:* $0 \leq z \leq 800$.

aim to give a unique (coordinate independent) definition of relative velocity between test particle and observer as measured along the observer's line-of-sight in generic pseudo-Riemannian spacetime. In doing this, the test particle is considered as a luminous object, otherwise, if it is not, we assume that a light source is attached to it, which has neither mass nor volume. Then, extending those geometrical ideas of well-known kinematic spectral shift rule to infinitesimal domain, we try to catch this effect by building a series of infinitesimally displaced shifts and then sum over them in order to find the proper answer to the problem that we wish to address. Thereby, the general equation (22) follows from a series of infinitesimal stretching of the proper space scale factor in Riemannian spacetime, whereas the path of a luminous source appears nowhere, thus this equation does not relate to the special choice of transport path. A resulting general expression (39) of relative velocity of test particle (a luminous source) as measured along the observer's line-of-sight in generic pseudo-Riemannian spacetime is utterly distinct from a familiar Doppler velocity. In particular case only, when adjacent observers are being in free fall and populated along the null geodesic, the relative velocity of luminous source is reduced to global Doppler velocity (38) as studied by Synge. We discuss the implications for several instructive cases of the Minkowski metric, the test particle and observer at rest in an arbitrary stationary metric, the uniform gravitational field, the rotating reference frame, the Schwarzschild metric, the Kerr-type metrics, and the spatially homogeneous and isotropic Robertson-Walker spacetime of standard cosmological model. The last case leads to cosmological consequence that resulting *kinetic* recession velocity of a distant astronomical object is always subluminal even for large redshifts of order one or more and, thus, it does not violate the fundamental physical principle of *causality*. This provides a new perspective to solve startling difficulties, which the conventional scenario of standard cosmological model presents. We calculate the measure, $\zeta(z, \dot{L})$, of carrying away of a galaxy at redshift $z$ by the expansion of space, such that it declines in magnitude to the $\zeta(25) = 0.034$, and then up to $\zeta(800) = 1.16 \times 10^{-3}$. This proves that cosmological expansion of a flat 3D–space is fundamentally different from a kinematics of galaxies moving in a non-expanding flat 3D-space.


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# Appendices

## Appendix A A reappraisal of the `standard´ kinematic interpretation

Finally, it is our purpose to give a reappraisal of the `standard´ kinematic interpretation. One source of alternative understanding of a complex problem is the `standard´ kinematic interpretation that cosmological redshift of distant galaxy is the recession effect of the accumulation of a series of infinitesimal Doppler shifts due to infinitesimal relative velocities of the Hubble flow along the line of sight (Bunn & Hogg, 2009, Chodorowski, 2011, Grøn & Elgarøy, 2007, Padmanabhan, 1993, Peacock, 1999, 2008, Peebles, 1993, Whiting, 2004). Within the `stretching of space´ point of view, one assumes that an observer at the origin at the present epoch time measures the redshift of a galaxy at some comoving distance. Consider a light ray that travels from a galaxy to this observer, both of whom are at rest in comoving coordinates. Imagine a family of comoving observers along the path of light ray, each of whom measures the frequency of light ray as it goes by. It was assumed that each observer is close enough to his neighbor so that we can accommodate them both in one inertial reference frame and use SR to calculate the change in frequency from one observer to the next. If adjacent observers are separated by the infinitesimal proper distance $\delta L$, then their relative velocity in this frame is $\delta v = H\delta L$. This infinitesimal recessional velocity should cause a fractional shift given by the non-relativistic Doppler formula:

$$\frac{\delta\nu}{\nu} = -\frac{H\delta L}{c} = -H\delta t. \tag{65}$$

And hence, as it was concluded, the relation (65) for redshift, by means of $H = \dot{R}/R$, becomes $\delta\nu/\nu = -\delta R/R$. This integrates to give the main result of expansion scenario that the frequency decreases in inverse proportion to the scale factor, $\nu \propto 1/R$.

However, a hard look at the basic relation (65) reveals the following three objections, which together constitute a whole against the claim.

(i) The equation (65) would lead to the relation $\delta\nu/\nu = -\delta R/R$ if, and only if, $\dot{L}(t,z) = c$, i.e. when galaxy situates on the Hubble sphere: $L = L_H \equiv c/H$, where $L_H$ is the Hubble length. But in general case of $\dot{L}(t,z) \neq c$ $(L \neq L_H)$, it was in conflict because the infinitesimal time interval $\delta t' = \delta L/c$ does not equal to the infinitesimal epoch time interval $\delta t = \delta L/\dot{L}$ $(\delta t'/\delta t = \dot{L}/c)$:

$$\frac{H\delta L}{c} = \frac{1}{R}\left(\frac{dR}{dt}\delta t\right)\frac{\delta L}{c\delta t} = \frac{\dot{L}(t,z)}{c}\frac{\delta R}{R}, \tag{66}$$

and hence (65) leads to

$$\frac{\delta\nu}{\nu} = -\frac{\dot{L}(t,z)}{c}\frac{\delta R}{R}. \tag{67}$$

This can readily be integrated to give

$$\delta \ln\nu = \delta\left[\ln\left(\frac{1}{R}\right)^{\dot{L}/c}\right] + \frac{\delta\dot{L}}{c}\ln R. \tag{68}$$

As it is clearly seen, (68) is utterly distinct from the simple behavior of $\nu \propto 1/R$.

(ii) If adjacent observers (i+1) and (i), separated by the infinitesimal proper distance $\delta L_i$, are situated at the distances $(L_i + \delta L_i)$ and $(L_i)$ from the observer at the origin at the present epoch, then not an infinitesimal difference in their velocities $\delta\beta_i$, but their SR relative velocity $\delta\beta_i^r$ should contribute to an infinitesimal Doppler shift. That is,

$$\frac{\delta\nu_i}{\nu_i} = -\delta\beta_i^r = -\frac{\delta\beta_i}{1-\beta_{i+1}\beta_i} = -\frac{H\delta L_i/c}{1-[H(L_i+\delta L_i)/c]\,(HL_i/c)} \approx -\frac{H\delta L_i/c}{1-(HL_i/c)^2}, \tag{69}$$


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the integration of which along path of light ray yields the Doppler shift:

$$\frac{\nu_1}{\nu} = \sqrt{\frac{1+\dot{L}/c}{1-\dot{L}/c}}, \tag{70}$$

which has nothing to do with the main result of expansion scenario $\nu \propto 1/R$. Moreover, the equation (70) limits the values of $\dot{L}$ to $\dot{L} \leq c$, which shows that it is incorrect to use $\dot{L}$ in equation (65).

(iii) As we have seen from the Fig. 3, the $\zeta(z)$ declines in magnitude to zero, which once more shows that it is incorrect to use in (65) the `proper´ recession velocity $\dot{L}$.

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