On the new opportunity of solving the Ambartsumian's functional equation in the theory of radiative transfer

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Abstract

A new approach to solving of the Ambartsumian's functional equation is presented. Its application is illustrated on the two classical cases: a) in the case of a plane-parallel semi-infinite medium with monochromatic anisotropic scattering and b) the simple one-dimensional problem of diffuse reflection of radiation from the scattering-absorbing semi-infinite medium in the case of the general law of redistribution of radiation by frequencies. The desired: a) azimuthal harmonics of reflection function dependent on two angular variables are explicitly expressed through the according eigenfunctions of one angle variable and b) diffuse reflection function of two frequency variables also expressed through a system of according eigenfunctions which have only one frequency variable. This does not require the use of any simplifying assumptions or special decompositions of the characteristics of elementary act of scattering: a) of scattering indicatrix and b) of the redistribution function of radiation by frequencies.

Keywords:radiative transfer, diffuse reflection problem, Ambartsumian's nonlinear functional equation, eigenfunctions and eigenvalues problem

1. Introduction and purpose of the work

The problem of diffuse reflection of radiation from a plane-parallel semi-infinite medium is one of the most important classical standard problems of theoretical astrophysics and, in particular, the theory of radiative transfer in scattering-absorbing media. It is widely used both in the interpretation of luminescence: planetary and stellar atmospheres, cosmic nebulae and various Space gas and dust complexes, and in the problems of optics of the Earth's atmosphere and ocean, the vegetation cover of the earth, as well as in the physics of nuclear reactors and radiation protection from ionizing radiation. The nonlinear integral equation for the direct determination of the diffuse reflection function from a semi-infinite medium in case of monochromatic scattering was obtained by introducing into the radiative transfer theory the so-called "Ambartsumian's principle of invariance and the method of addition of layers" (Ambartsumian, 1942b, 1943a,b, 1944). In the case of anisotropic scattering for the azimuthal harmonics of the reflection coefficient, the Ambartsumian's functional equation has the form (see, for example, Sobolev (1972) p. 51 or Yanovitskij (1995) p. 71).

$$\begin{gathered}(\mu+\mu')\,\rho^m(\mu,\mu')=\tfrac{\lambda}{4}\chi^m(-\mu,\mu')+\tfrac{\lambda}{2}\mu'\int\limits_0^1\chi^m(\mu,\mu'')\,\rho^m(\mu'',\mu')\,d\mu''\\ +\tfrac{\lambda}{2}\mu\int\limits_0^1\rho^m(\mu,\mu'')\,\chi^m(\mu'',\mu')\,d\mu''+\lambda\mu\mu'\int\limits_0^1\int\limits_0^1\rho^m(\mu,\mu''')\,\chi^m(\mu''',-\mu'')\,\rho^m(\mu'',\mu')\,d\mu''d\mu'''\end{gathered}\tag{1}$$

Here: $\rho^m(\mu,\mu')$ is the azimuthal harmonic of the desired brightness coefficient, $\chi^m(\mu,\mu')$ is the azimuthal harmonic of the scattering indicatrix, λ is the single scattering albedo of the quantum in the elementary act of scattering, μ' and μ are respectively the cosines of the angles of incidence and reflection of radiation from a semi-infinite medium with respect to the normal to its inner boundary. In the case of a one-dimensional semi-infinite medium, when there is a redistribution of radiation by frequencies, a similar equation was obtained by Sobolev (1955), which in generally accepted notations is written in the form:

$$\begin{gathered}\tfrac{2}{\lambda}\left[\alpha(x)+\alpha(x')\right]\rho(x,x')=r(x,x')+\int\limits_{-\infty}^{+\infty}r(x,\ x'')\,\rho(\ x'',\ x')\,dx''\\ +\int\limits_{-\infty}^{+\infty}\rho(x,x''')\,r(x''',x')\,dx'''+\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\rho(x,\ x''')\,r(x''',\ x'')\,\rho(x'',\ x')\,dx''dx'''\end{gathered}\tag{2}$$

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Here: $\rho\left(x,x'\right)dx$ is the probability of diffuse reflection of the quantum from the medium in the $(x,\ x{+}dx)$ range of dimensionless frequencies, when entering a medium quantum had a frequency of x', $r\left(x''',\ x''\right)$ is the frequency redistribution function of radiation at an elementary act of scattering, $\alpha(x)$ is the absorption coefficient profile.

Traditionally, when solving equations (1) and (2), various expansions of the characteristics of the elementary act of scattering are used: in the case of anisotropic scattering, the scattering indicatrix is decomposed into Legendre polynomials (Ambartsumian, 1942a, 1943a, Chandrasekhar, 1950), then the m-th azimuthal harmonic of the scattering indicatrix takes the form

$$\chi^m\left(\mu,\mu''\right)=\sum_{i=m}^{N}c_i^m P_i^m\left(\mu\right)P_i^m\left(\mu'\right), \tag{3}$$

where $P_i^m\left(\mu\right)$ are the adjunctive Legendre functions. In the case of non-coherent scattering, for example, the bilinear decomposition of the frequency redistribution function by its eigenfunctions - $\alpha_j\left(x\right)$ (Engibaryan, 1971, Gevorkyan & Khachatryan, 1985, Gevorkyan et al., 1975, Khachatrian et al., 1991) was used

$$r\left(x,x'\right)=\sum_j A_j\alpha_j\left(x\right)\alpha_j\left(x'\right). \tag{4}$$

Thus, the kernels of nonlinear integral equations (1) and (2) approximately became degenerate, which made it possible to obtain the solution of the main problem explicitly, through the so-called of Ambartsumian's auxiliary functions $\varphi_i^m\left(\mu\right)$ and $\varphi_j\left(x\right)$ having a smaller number of variables:

$$\left(\mu+\mu'\right)\rho^m\left(\mu,\mu'\right)=\frac{\lambda}{4}\sum_{i=m}^{N}c_i^m\left(-1\right)^{i+m}\varphi_i^m\left(\mu\right)\varphi_i^m\left(\mu'\right), \tag{5}$$

$$\left[\alpha\left(x\right)+\alpha\left(x'\right)\right]\rho\left(x,x'\right)=\frac{\lambda}{2}\sum_j A_j\varphi_j\left(x\right)\varphi_j\left(x'\right), \tag{6}$$

effectively reducing their finding to the corresponding systems of functional equations:

$$\varphi_i^m\left(\mu\right)=P_i^m\left(\mu\right)+\frac{\lambda}{4}\mu\sum_{j=m}^{N}c_j^m\left(-1\right)^{i+j}\varphi_j^m\left(\mu\right)\int\limits_0^1\frac{P_i^m\left(\mu'\right)\varphi_j^m\left(\mu'\right)}{\mu+\mu'}d\mu', \tag{7}$$

$$\varphi_m\left(x\right)=\alpha_m\left(x\right)+\frac{\lambda}{2}\sum_k A_k\varphi_k\left(x\right)\int\limits_{-\infty}^{+\infty}\frac{\varphi_k\left(x'\right)\alpha_m\left(x'\right)}{\alpha\left(x\right)+\alpha\left(x'\right)}dx'. \tag{8}$$

However, in cases where the representation of real or model indicatricis, as well as the functions of redistribution of radiation by frequencies, require taking into account a significantly large number of terms in expansions of types (3) and (4), the task will be significantly more complicated. Indeed, for example, with a real cloud indicatrix (Smoktiy & Anikonov, 2008), 229 is required in the decomposition (3), and with its approximate replacement by the Henyey-Greenstein phase function type model, 152 terms are required (see below, Figure 1, from the book Smoktiy & Anikonov (2008) Fig. 4.1.1).

In the case of non-coherent scattering, the construction of a general representation of the redistribution function (4), taking into account various physical factors, is already a rather time-consuming task (Arutyunyan, 1991, Gevorkyan & Khachatryan, 1985, Khachatrian et al., 1991):

$$r_V\left(x,x'\right)=\begin{cases} r_{III}\left(x,x'\right), & \sigma_j=0\\ r_{II}\left(x,x'\right), & \sigma_i=0\\ r_I\left(x,x'\right), & \sigma_j=\sigma_i=0\end{cases},\quad r_V\left(x,x'\right)=\sum_k\frac{\beta_{2k}\left(x,\sigma_i,\sigma_j\right)\beta_{2k}\left(x',\sigma_i,\sigma_j\right)}{\lambda_k\left(\sigma_j\right)},$$

$$\beta_{2k}\left(x,\sigma_i,\sigma_j\right)=\sum_m\gamma_{km}\left(\sigma_j\right)\alpha_{2m}\left(x,\sigma_i\right),\quad \alpha_k\left(x,\sigma_i\right)=\frac{\sigma_i}{\pi}\int\limits_{-\infty}^{+\infty}\frac{\alpha_k\left(t\right)dt}{\left(x-t\right)^2+{\sigma_i}^2}, \tag{9}$$

not even to mention the number of necessary terms of the expansion to achieve a certain predetermined accuracy of the desired solution of the diffuse reflection problem. In the case of incoherent scattering, it is

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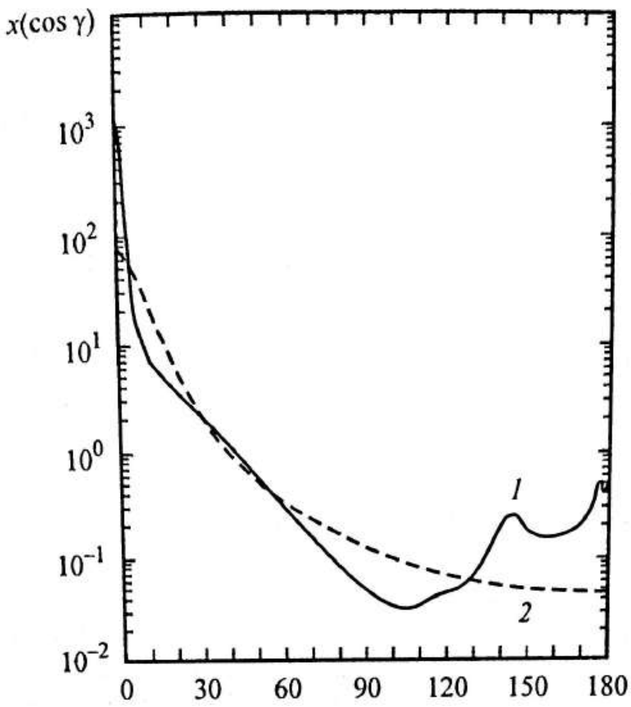

Figure 1. The scattering indicatrix - 1, $\chi\left(\cos\gamma\right)$ for the cloud model (solid line) and the model indication $\chi_{H-G}\left(\cos\gamma\right)$ Henyey-Greenstein- 2 (dashed line) approximating it. On the ordinate axis, the indicatrix $\chi(\cos\gamma)$, and on the abscissa axis, the scattering angle γ in degrees.

also important to mention the method of directly searching for the reflection function in the form Gevorkyan & Khachatryan (1985), Khachatrian et al. (1991):

$$\rho\left(x,x'\right) = \sum_i\sum_k \rho_{ik}\frac{\alpha_i\left(x\right)\alpha_k\left(x'\right)}{\alpha\left(x\right)},\quad \rho_{ik} = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}\rho\left(x,x'\right)\frac{\alpha_i\left(x\right)\alpha_k\left(x'\right)}{\alpha\left(x'\right)}dx'dx. \tag{10}$$

As we can see here, the desired solution of the problem of diffuse reflection $\rho\left(x,x'\right)$ is directly represented by a "double bilinear" series of eigenfunctions $\alpha_i\left(x\right)$ of the general frequency redistribution law $r\left(x,x'\right)$, where the coefficients ρ_{ik} are from the corresponding system of nonlinear equations. The authors of the work Gevorkyan & Khachatryan (1985) point to the effectiveness of this method of searching for $\rho\left(x,x'\right)$ in comparison with the previous one in terms of numerical calculations, motivating this by the presence here of a criterion that ensures the proper accuracy of solving the problem. However, similar to the previous method, it is also necessary to pre-construct the expansion $r\left(x,x'\right)$.

A natural question arises: is it possible to solve the problem of diffuse reflection, as before, to be reduced to finding auxiliary functions of a smaller number of variables, but at the same time to do without decomposing the characteristics of the elementary act of scattering? From the foregoing, it already follows the expediency of such a formulation of the question, since the search for the necessary decomposition of the nuclei of the elementary act of scattering is an additional and not always simple task. Moreover, the primary physical features of the characteristics of a single scattering act after each new scattering become smoother due to the next integration procedure. As a result, the final characteristics of the fields of multiple scattered radiation, of course, will have a smoother behavior, which will greatly simplify their description. Thus, from physical considerations, it follows that, in the form of corresponding expansions, it is more expedient to directly find the resulting radiation fields instead of the nuclei of the elementary scattering act. Obviously, under equal conditions, the corresponding series for $\rho^m\left(\mu,\mu'\right)$ and $\rho\left(x,x'\right)$ will be shorter than (3) and (4). The purpose of the presented work is the analytical implementation of this possibility, using the example of the above two cases described by functional equations (1) and (2).

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2. Formulation and solution of the problem

a) Anisotropic monochromatic scattering. On the right side of equation (1) we enter the notation

$$K^m\left(\mu, \mu'\right) \equiv \left(\mu+\mu'\right) \rho^m\left(\mu, \mu'\right), \tag{11}$$

at the same time

$$\rho^m\left(\mu, \mu'\right)=\rho^m\left(\mu', \ \mu\right) \Rightarrow K^m\left(\mu, \mu'\right)=K^m\left(\mu', \ \mu\right). \tag{12}$$

It is not difficult from equation (1), taking into account (11), to obtain for the introduced symmetric and positive function $K^m\left(\mu, \mu'\right)$ a functional equation

$$\begin{gathered}K^m\left(\mu, \mu'\right)=\frac{\lambda}{4} \chi^m\left(-\mu, \mu'\right)+\frac{\lambda}{2} \mu' \int\limits_0^1 \chi^m\left(\mu, \mu''\right) \frac{K^m\left(\mu'', \mu'\right)}{\mu''+\mu'} d \mu''+\frac{\lambda}{2} \mu \int\limits_0^1 \frac{K^m\left(\mu, \mu''\right)}{\mu+\mu''} \chi^m\left(\mu'', \mu'\right) d \mu''+ \\ \lambda \mu \mu' \int\limits_0^1 \int\limits_0^1 \frac{K^m\left(\mu, \mu'''\right)}{\mu+\mu'''} \chi^m\left(\mu''', -\mu''\right) \frac{K^m\left(\mu'', \mu'\right)}{\mu''+\mu'} d \mu'' d \mu'''\end{gathered}. \tag{13}$$

Let us pose the standard problem of finding eigenvalues and eigenvectors of a symmetric integral operator $\int\limits_0^1 K^m\left(\mu, \mu'\right) \ldots d \mu'$:

$$\nu_l^m \beta_l^m(\mu)=\int\limits_0^1 K^m\left(\mu, \mu'\right) \beta_l^m\left(\mu'\right) d \mu', \quad \int\limits_0^1 \beta_n^m(\mu) \beta_l^m\left(\mu'\right) d \mu'=\delta_{l n}. \tag{14}$$

Since the unknown in (14) kernel $K^m\left(\mu, \mu'\right)$ is given only by means of its nonlinear functional equation (13), then to find the appropriate eigenfunctions act with the integral operator $\int\limits_0^1 \ldots \beta_l^m\left(\mu'\right) d \mu'$ on equation (13) and take into account (14). Then, in the subintegral expressions of the resulting ratio, we take into account the possibility of an approximate representation, with any predetermined accuracy, the desired positive and symmetric kernel $K^m\left(\mu, \mu'\right)$ by means of a bilinear series of its eigenfunctions

$$K^m\left(\mu, \mu'\right)=\sum_n \nu_n^m \beta_n^m(\mu) \beta_n^m\left(\mu'\right). \tag{15}$$

After simple calculations, we come to a system of equations:

$$\nu_l^m \beta_l^m(\mu)=\frac{\lambda}{4} Z_l^m(\mu)+\frac{\lambda}{2} \sum_n \nu_n^m D_{n l}^m(\mu)+\lambda \sum_n \sum_k \nu_n^m \nu_k^m V_{n k l}^m(\mu). \tag{16}$$

Here, the quantities $Z_l^m(\mu)$, $D_{n l}^m(\mu)$, $V_{n k l}^m(\mu)$ obviously include the desired eigenfunctions $\beta_l^m(\mu)$:

$$Z_l^m(\mu)=\int\limits_0^1 \chi^m\left(-\mu, \mu'\right) \beta_l^m\left(\mu'\right) d \mu', \quad w_{n l}^m\left(\mu''\right)=\beta_n^m\left(\mu''\right) \int\limits_0^1 \frac{\beta_n^m\left(\mu'\right)}{\mu''+\mu'} \beta_l^m\left(\mu'\right) \mu' d \mu',$$

$$D_{n l}^m(\mu)=\int\limits_0^1 \chi^m\left(\mu, \mu''\right) w_{n l}^m\left(\mu''\right) d \mu''+\mu \beta_n^m(\mu) \int\limits_0^1 \frac{\beta_n^m\left(\mu''\right)}{\mu+\mu''} d \mu'' \int\limits_0^1 \chi^m\left(\mu'', \mu'\right) \beta_l^m\left(\mu'\right) d \mu',$$

$$V_{n k l}^m(\mu)=\mu \beta_n^m(\mu) \int\limits_0^1 \frac{\beta_n^m\left(\mu'''\right)}{\mu+\mu'''} d \mu''' \int\limits_0^1 \chi^m\left(-\mu''', \mu''\right) w_{k l}^m\left(\mu''\right) d \mu''. \tag{17}$$

To determine the unknown in (16) eigenvalues ν_l^m , it is not difficult from (16), taking into account (17) and the orthogonality condition (14), to obtain a system

$$\nu_l^m=\frac{\lambda}{4} b_l^m+\frac{\lambda}{2} \sum_n \nu_n^m c_{n l}^m+\lambda \sum_n \sum_k \nu_n^m \nu_k^m f_{n k l}^m, \tag{18}$$

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where the values b_l^m , c_{nl}^m , f_{nkl}^m in turn depend on the desired eigenfunctions:

$$b_l^m = \int\limits_0^1 \int\limits_0^1 \beta_l^m(\mu)\,\chi^m(-\mu,\mu')\,\beta_l^m(\mu')\,d\mu' d\mu, \quad c_{nl}^m = \int\limits_0^1 \int\limits_0^1 \beta_l^m(\mu)\,\chi^m(\mu,\mu'')\,w_{nl}^m(\mu'')\,d\mu'' d\mu,$$

$$f_{nkl}^m = \int\limits_0^1 \int\limits_0^1 w_{nl}^m(\mu''')\,\chi^m(\mu''',-\mu'')\,w_{kl}^m(\mu'')\,d\mu'' d\mu'''. \tag{19}$$

Expressions (16)-(19) together represent a system of equations for the self-consistent determination of the desired eigenfunctions and eigenvalues of the kernel $K^m(\mu,\mu')$. The solution of the initial problem of determining the azimuthal harmonics of the brightness coefficient is given by expression

$$\rho^m(\mu,\mu') = \frac{\sum_n \nu_n^m \beta_n^m(\mu)\,\beta_n^m(\mu')}{\mu+\mu'}. \tag{20}$$

b) A one-dimensional medium in the case of the general of the frequency redistribution. In equation (2) we introduce the notations:

$$K(x,x') \equiv \left[\alpha(x)+\alpha(x')\right]\rho(x,x'), \quad \varphi(x,x') \equiv \delta(x-x')+\rho(x,\ x'), \tag{21}$$

then it will take the form:

$$\frac{2}{\lambda}K(x,x') = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varphi(x,x''')\,r(x''',x'')\,\varphi(x'',x')\,dx''dx'''. \tag{22}$$

Hence, taking into account (21), for the value $K(x,x')$ we obtain a nonlinear integral equation:

$$\frac{2}{\lambda}K(x,x') = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[\delta(x-x''') + \frac{K(x,\ x''')}{\alpha(x)+\alpha(x''')}\right] r(x''',x'') \left[\delta(x''-x') + \frac{K(x'',\ x')}{\alpha(x'')+\alpha(x')}\right] dx''dx'''. \tag{23}$$

Let us pose the problem of finding the eigenfunctions and eigenvalues of the positive and symmetric kernels $K\ (x,x')$:

$$\nu_i\beta_i(x) = \int_{-\infty}^{+\infty} K(x,x')\,\beta_i(x')\,dx', \quad \int_{-\infty}^{+\infty} \beta_i(x)\,\beta_j(x)\,dx = \delta_{ij}, \tag{24}$$

The previously unknown nucleus $K\ (x,x')$ is given by means of its nonlinear functional equation (23). By influencing this equation with the operator $\int_{-\infty}^{+\infty}\ldots\beta_j(x)\,dx$ and taking into account the possibility of an approximate representation of a symmetric positive kernel (with an arbitrary given precision) through a bilinear series of its eigenfunctions

$$K(x,x') = \sum_j \nu_j \beta_j(x)\,\beta_j(x'), \tag{25}$$

after simple calculations, it is not difficult to obtain a system of nonlinear equations for the desired eigenfunctions $\beta_k(x)$

$$\frac{2}{\lambda}\nu_k\beta_k(x) = Z_k(x) + \sum_j \nu_j D_{jk}(x) + \sum_j\sum_i \nu_i\nu_j V_{jik}(x). \tag{26}$$

The quantities $Z_k(x)$, $D_{jk}(x)$, $V_{jik}(x)$ appearing here are determined by means of the searched eigenfunctions $\beta_k(x)$:

$$Z_k(x) = \int_{-\infty}^{+\infty} r(x,x')\,\beta_k(x')\,dx', \quad w_{lk}(x'') = \beta_l(x'') \int_{-\infty}^{+\infty} \frac{\beta_l(x')}{\alpha(x'')+\alpha(x')}\beta_k(x')\,dx',$$

$$D_{jk}(x) = \beta_j(x)\int_{-\infty}^{+\infty} \frac{\beta_j(x''')}{\alpha(x)+\alpha(x''')} Z_k(x''')\,dx''' + \int_{-\infty}^{+\infty} r(x,x'')\,w_{jk}(x'')\,dx'',$$

$$V_{jik}(x) = \beta_j(x)\int_{-\infty}^{+\infty} \frac{\beta_j(x''')}{\alpha(x)+\alpha(x''')}dx''' \int_{-\infty}^{+\infty} r(x''',x'')\,w_{ik}(x'')\,dx''. \tag{27}$$

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In the system (26), the values of the eigenvalues ν_k are not yet known. Influencing the same integral operator $\int_{-\infty}^{+\infty} \ldots \beta_j(x)\, dx$ to the system (26), taking into account the orthogonality of eigenfunctions, we obtain a nonlinear algebraic system for determining eigenvalues ν_k :

$$\frac{2}{\lambda}\nu_k = b_k + \sum_j \nu_j c_{jk} + \sum_j \sum_i \nu_i \nu_j f_{jik}. \tag{28}$$

The quantities unknown here b_k , c_{jk} , f_{jik}, as in (26), are expressed in terms of the desired eigenfunctions $\beta_i(x)$ of the problem (24) and are represented as:

$$b_k = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \beta_k(x)\, r(x, x')\, \beta_k(x')\, dx dx',$$

$$c_{jk} = 2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \beta_k(x''')\, r(x''', x'')\, w_{jk}(x'')\, dx'' dx''',$$

$$f_{jik} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} w_{jk}(x''')\, r(x''', x'')\, w_{ik}(x'')\, dx'' dx'''. \tag{29}$$

Thus, the one-dimensional problem of diffuse reflection of radiation from a semi-infinite scattering-absorbing medium under the general law of frequency redistribution, similar to the problem of anisotropic scattering, is reduced to a self-consistent joint solution of systems (26)-(29) for determining ν_k and $\beta_k(x)$, and then constructing the final solution in the form of

$$\rho(x, x') = \frac{\sum_j \nu_j \beta_j(x)\, \beta_j(x')}{\alpha(x) + \alpha(x')}. \tag{30}$$

3. Relationship with other methods

As mentioned above, the proposed approach to solving the problem of diffuse reflection will be more economical in comparison with the methods of decomposition of a single scattering act, since here the bilinear series is searched directly for quantities describing multiple scatterings. And the latter undoubtedly have a smoother and "integral" behavior than functions describing a single act of scattering. Due to the smoother behavior of these "resultant" radiation fields, it is natural to expect that the same accuracy in solving the original problem here will be achieved by a smaller number of terms of the bilinear expansion. That is, under equal conditions, the decomposition of the characteristics of the resulting field will be described by a smaller number of eigenfunctions than the number of auxiliary functions in the methods mentioned above. Another advantage is that it is no longer necessary to solve the additional problem of the decomposition of the characteristics of the elementary act of scattering. For a quantitative comparison of the effectiveness of the described two methods for solving the initial problem, it is very advisable to establish a two-way relationship between the quantities $\beta_i^m(\mu)$ and $\varphi_i^m(\mu)$, as well as between $\beta_i(x)$ and $\varphi_i(x)$.

a) Anisotropic monochromatic scattering. Comparison of solutions (5) and (20) gives

$$\sum_n \nu_n^m \beta_n^m(\mu)\, \beta_n^m(\mu') = \frac{\lambda}{4} \sum_i c_i^m (-1)^{i+m} \varphi_i^m(\mu)\, \varphi_i^m(\mu'). \tag{31}$$

Applying the condition of orthogonality of eigenfunctions $\beta_n^m(\mu)$, we obtain their connection with the Ambartsumian's auxiliary functions

$$\nu_n^m \beta_n^m(\mu) = \frac{\lambda}{4} \sum_i c_i^m (-1)^{i+m} \varphi_i^m(\mu)\, q_{in}^m, \quad q_{in}^m \equiv \int_0^1 \varphi_i^m(\mu')\, \beta_n^m(\mu')\, d\mu'. \tag{32}$$

The application of the condition of orthogonality of eigenfunctions in the first of the relations (32) and the subsequent consideration of the second leads to an explicit expression for determining eigenvalues ν_n^m

$$\nu_n^m = \frac{\lambda}{4} \sum_i c_i^m (-1)^{i+m} (q_{in}^m)^2. \tag{33}$$

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To find the unknown q_{in}^m appearing in (32) and (33), using its definition and explicit expression for $\beta_n^m(\mu)$ from (32) we obtain the relation

$$\nu_n^m q_{kn}^m = \frac{\lambda}{4}\sum_i c_i^m (-1)^{i+m} a_{ki}^m q_{in}^m, \tag{34}$$

where a_{ki}^m are given by

$$a_{ki}^m(\mu) = \int_0^1 \varphi_k^m(\mu)\,\varphi_i^m(\mu)\,d\mu. \tag{35}$$

From (34) and (33) we find the system

$$q_{kn}^m = \frac{\sum_i c_i^m (-1)^{i+m} a_{ki}^m q_{in}^m}{\sum_i c_i^m (-1)^{i+m} (q_{in}^m)^2}, \tag{36}$$

and to find the eigenfunctions $\beta_n^m(\mu)$ from (33) and (32), an explicit expression is obtained

$$\beta_n^m(\mu) = \frac{\sum_i c_i^m (-1)^{i+m} \varphi_i^m(\mu)\, q_{in}^m}{\sum_i c_i^m (-1)^{i+m} (q_{in}^m)^2}. \tag{37}$$

Feedback, i.e., the definition of $\varphi_i^m(\mu)$ when the eigenfunctions $\beta_n^m(\mu)$ are known, is obtained from the well-known definition of Ambartsumian's auxiliary functions

$$\varphi_i^m(\mu) = P_i^m(\mu) + 2\mu(-1)^{i+m}\int_0^1 P_i^m(\mu')\,\rho^m(\mu',\mu)\,d\mu'. \tag{38}$$

Substituting solution (20) here, we come to the expression

$$\varphi_i^m(\mu) = P_i^m(\mu) + 2\mu(-1)^{i+m}\sum_n \nu_n^m \beta_n^m(\mu)\, Q_{ni}^m(\mu), \tag{39}$$

where is

$$Q_{ni}^m(\mu) \equiv \int_0^1 \frac{\beta_n^m(\mu')\, P_i^m(\mu')}{\mu+\mu'}d\mu'. \tag{40}$$

b) One-dimensional medium under the general law of redistribution of radiation by frequencies. Let's compare solutions (6) and (30), similar to the previous paragraph "a", we will have

$$\sum_j \nu_j \beta_j(x)\,\beta_j(x') = \frac{\lambda}{2}\sum_k A_k \varphi_k(x)\,\varphi_k(x'), \tag{41}$$

then using the orthogonality condition of eigenfunctions $\beta_j(x)$ will give the expressions:

$$\nu_j\beta_j(x) = \frac{\lambda}{2}\sum_k A_k\varphi_k(x)\,q_{kj}, \quad q_{kj} \equiv \int_{-\infty}^{+\infty} \varphi_k(x')\,\beta_j(x')\,dx', \tag{42}$$

taking into account the orthogonality condition in the first relation (42), in turn, will give

$$\nu_j = \frac{\lambda}{2}\sum_k A_k (q_{kj})^2. \tag{43}$$

From (42) and (43) for the eigenfunctions $\beta_j(x)$, we finally get an explicit expression

$$\beta_j(x) = \frac{\sum_k A_k\varphi_k(x)\,q_{kj}}{\sum_k A_k (q_{kj})^2}. \tag{44}$$

In the ratios (42) and (43), the magnitude of q_{kj} is still unknown. To determine it, first by substituting the first ratio from (42) to the second, we get the formulas:

$$\nu_j q_{mj} = \frac{\lambda}{2}\sum_k A_k a_{mk} q_{kj}, \quad a_{mk} \equiv \int_{-\infty}^{+\infty} \varphi_m(x)\,\varphi_k(x)\,dx, \tag{45}$$

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and then, taking into account (43), we come to the system of equations

$$q_{mj} = \frac{\sum_k A_k a_{mk} q_{kj}}{\sum_k A_k \left(q_{kj}\right)^2}. \tag{46}$$

As a result, if the auxiliary functions of Ambartsumian are determined by the method of decomposition of the frequency redistribution function, then the transition to the eigenfunctions of the method proposed in this work is carried out by solving the system (46) and explicit expressions (43), (44). To derive the feedback-finding of auxiliary functions $\varphi_m(x)$ through the previously known eigenfunctions $\beta_j(x)$ and eigenvalues ν_j, recall their definition

$$\varphi_m(x) = \alpha_m(x) + \int\limits_{-\infty}^{+\infty} \rho\left(x, x'\right) \alpha_m\left(x'\right) dx'. \tag{47}$$

After substituting in (47) the solution (30), the final expressions will be obtained

$$\varphi_m(x) = \alpha_m(x) + \sum_j \nu_j \beta_j(x) Q_{jm}(x), \quad Q_{jm}(x) = \int\limits_{-\infty}^{+\infty} \frac{\beta_j\left(x'\right) \alpha_m\left(x'\right)}{\alpha(x) + \alpha\left(x'\right)} dx'. \tag{48}$$

The presence of (33), (36), (37) together with (39), (40), also (43), (44), (46) together with (48) allow in both problems "a" and "b" to evaluate and compare the accuracy of the results obtained by different methods.

4. The general scheme of the organization of calculations

To calculate the desired eigenfunctions and eigenvalues above, two pairs of systems were obtained: the first pair - (16), (18) in the anisotropic scattering problem, and the second (26), (28) in the incoherent scattering problem. Each pair is to be calculated jointly, in a self-consistent manner – a certain system of orthonormal functions is taken as a zero approximation of the desired eigenfunctions (for example, in the anisotropic scattering problem, the attached Legendre functions, and in the incoherent scattering problem, Hermite polynomials). Obviously, the general structure of systems (16) and (26), as well as (18) and (28) with an accuracy of factors of type $\lambda/2$ is identical, so the scheme for their calculation is the same. In the problem of incoherent scattering, for example, the choice of the initial system of orthonormal functions will be "given" zero approximations of the desired quantities - $\left[\beta_k(x)\right]^{(0)}$.With their help, according to formulas (29), the zero approximation: $\left[b_k\right]^{(0)}$, $\left[c_{jk}\right]^{(0)}$, $\left[f_{jik}\right]^{(0)}$of the quantities is calculated. By substituting the latter in the right side of the relation (28), as well as taking here $\left[\nu_k\right]^{(0)} \equiv \left[b_k\right]^{(0)}$ as a zero approximation of eigenvalues, the subsequent first approximation - $\left[\nu_k\right]^{(1)}$ for eigennumbers in the left side of (28), will be obtained. Then, by means of formulas (27) using $\left[\beta_k(x)\right]^{(0)}$, the zero approximations of the functions $\left[Z_k(x)\right]^{(0)}$, $\left[D_{jk}(x)\right]^{(0)}$, $\left[V_{jik}(x)\right]^{(0)}$ are calculated. By substituting the values of the calculated functions in the right side (26), using the calculated first approximation - $\left[\nu_k\right]^{(1)}$, the values of the first approximation - $\left[\beta_k(x)\right]^{(1)}$of eigenfunctions are obtained. Then, the values $\left[\beta_k(x)\right]^{(n)}$ are taken as the initial approximation of the eigenfunctions and the entire described cycle is repeated. Calculations on such cycles of successive approximations $\left[\beta_k(x)\right]^{(n)}$ are repeated until the number n is reached, which gives the necessary accuracy. The same general scheme of organization of successive stages of calculations is illustrated in Fig. 2, in relation to the problem of determining the azimuthal harmonics of the brightness coefficient in the case of anisotropic scattering.

5. Conclusion

The paper presents a new possibility of solving Ambartsumian's functional equation in the problem of diffuse reflection of radiation from a semi-infinite scattering-absorbing medium. The expediency and effectiveness of the proposed approach follows from the physically obvious fact that in the process of multiple scattering of primary radiation - "diffusion" of quanta (or particles) in the medium, with each subsequent scattering act, the intensity of the formed radiation field (or the phase density of particles) becomes a mathematically smoother quantity. Therefore the problem of representing the resulting field through a bilinear series of "own" eigenfunctions is simpler and more efficient, compared to a similar problem of a single

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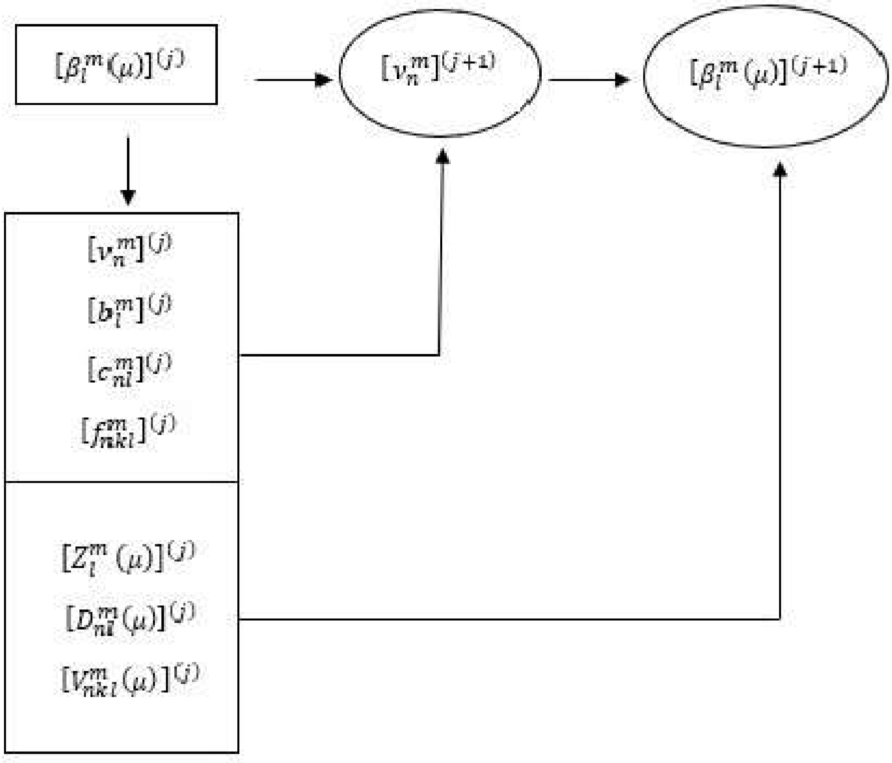

Figure 2. Diagram of sequential steps, self-consistent joint calculations of eigenvalues and eigenfunctions.

act of scattering. The method is analytically illustrated by two standard classical problems: finding the azimuthal harmonics of the reflection coefficient of monochromatic radiation from a semi-infinite medium with anisotropic scattering and diffuse reflection of radiation from a one-dimensional semi-infinite medium with the general law of frequency redistribution of radiation. Explicit expressions of solutions to the considered problems of diffuse reflection, depending on two independent variables, through the corresponding eigenfunctions of one independent variable, are obtained. To find the latter, as well as the corresponding eigenvalues in each of these cases, a pair of two systems of nonlinear equations is derived: functional and algebraic.

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