Separation of Frequency Variables in the Problem of Diffuse Reflection from a Semi-Infinite Medium under Isotropic Scattering with a General Law of Radiation Redistribution by Frequencies

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Abstract

In the presented work, with the help of a new approach previously proposed by the author, the problem of diffuse reflection of radiation from a plane-parallel semi-infinite medium under isotropic scattering in the case of the general law of radiation redistribution by frequencies is solved. Resulting solution implements the possibility of separating a pair of independent variables (namely, the frequency and direction of the quantum) of entering the medium quanta from the same pair of exiting quanta. The advantage of this approach in relation to the known methods is that the separation of the explanatory variables is achieved without the need to solve the additional problem of separation or any special representation of the characteristics of a single act of scattering. Here, in an expanded form is sought namely the solution of the modified initial problem, instead of a preliminary decomposition of the characteristics of a single act of scattering. As a result, the unknown function of the four explanatory variables is expressed explicitly through a system of auxiliary functions that depend on only two variables. For this purpose, a problem for eigenvalues and eigenfunctions is formulated for a specially selected and previously unknown kernel. Bilateral relationships between the solutions of the new and traditionally used methods are obtained, which makes it possible to directly compare their accuracy and efficiency. The general scheme of the organization of calculations is also discussed.

Keywords: *radiative transfer, diffuse reflection problem, separation of variables, nonlinear Ambartsumian's functional equation, eigenfunctions and eigenvalues problem*

1. Introduction and purpose of the work

The problem of diffuse reflection of radiation (or particles) from a semi-infinite plane-parallel medium is one of the classical problems of theoretical astrophysics. It is widely used to interpret the brightness and spectra of space objects and cosmic media where there is multiple interactions between radiation and matter. For example, planetary and stellar atmospheres, the interstellar medium, various kinds of space gas and dust complexes, as well as the optics of the Earth's atmosphere and ocean, the engineering problems of nuclear reactors, as well as radiation protection from ionizing radiation, etc. The solution of the diffuse reflection problem depends on many independent variables that describe the parameters of diffusing photons (or particles) as they enter the medium and then when they exit it after multiple scatterings. Therefore, when solving such problems, they traditionally try to separate independent variables from each other, as a result, reducing the problem to the construction of auxiliary functions of a smaller number of independent variables. The introduction of the so-called "Ambartsumian's principle of invariance" (Ambartsumian, 1942, 1943a,b, 1944a,b,c)made it possible to obtain a functional equation for the direct finding of the diffuse reflection function. In contrast to the traditional way of solving the integro differential or integral transfer equations, there was no need to consider the intensity of the radiation coming out of the medium together with the field inside the medium, i.e., it was possible to limit oneself to the analysis of the processes of absorption and re-emission of quanta only at the boundary of the medium. In solving the problem of diffuse reflection in the case of monochromatic and isotropic scattering, the exact separation of angular variables in this way was achieved in the work Ambartsumian (1942) and see also Sobolev (1963), i.e., the desired diffuse reflection function $\rho(\mu, \mu')$, which depends on two angular variables, was clearly expressed through

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an auxiliary function, $\varphi(\mu)$ of only one angular variable:

$$\rho(\mu,\mu') = \frac{\lambda}{2\mu'}\frac{\varphi(\mu)\varphi(\mu')}{\frac{1}{\mu}+\frac{1}{\mu'}}, \qquad \varphi(\mu) = 1 + \int_0^1 \rho(\mu,\mu')\,d\mu', \tag{1}$$

$$\varphi(\mu) = 1 + \frac{\lambda}{2}\mu\varphi(\mu)\int_0^1 \frac{\varphi(\mu')}{\mu+\mu'}d\mu'. \tag{2}$$

Here: $\rho(\mu,\mu')$ is the probability density that the quantum falling on the boundary of the medium from the direction μ', inside the solid angle $2\pi d\mu'$ after arbitrary wanderings in the medium will leave it in the direction of μ, inside the solid angle $2\pi d\mu$ (μ' and μ are cosines of the angles of incidence and diffuse reflection at the boundary of the medium in relation to the normal to its boundary, respectively), λ is the probability of the quantum surviving in the elementary act of scattering.

A generalization of the problem in the case of anisotropic scattering was carried out in the paper (Ambartsumian, 1943a, 1944a). In order to achieve separation of angular variables in the problem of diffuse reflection in the case of anisotropic scattering, an additional problem of separation of angular variables in a single act of scattering was posed and solved. This was done by means of a special representation of the scattering indicatrix in the form of a series over Legendre polynomials (Ambartsumian, 1941, § 8-9). In the case of incoherent scattering, it was necessary to separate the frequency variables. Such a property is naturally manifested in the case of so-called "complete redistribution of radiation by frequencies" or "completely incoherent scattering" takes place during an elementary act of scattering (see for example: Ivanov, 1973, Sobolev, 1963). In the case of the general law of redistribution of radiation by frequencies or the so-called "partially incoherent scattering", the separation of frequency variables in the diffuse reflection problem is achieved by representing the redistribution function in a unit act of scattering as a bilinear series over a pre-selected system of eigenfunctions (Engibaryan, 1971, Engibaryan & Nikogosyan, 1972). Further, it was noted that in the case of the general law of redistribution of radiation by frequencies, in particular case of isotropic scattering, the property of separation of angular variables naturally already follows from the isotropy of a single act of scattering (Pikichian, 1978, 1980). At the same time, since the frequency variables are not separated in the unit act of scattering, they are not separated also in multiple scatterings, i.e. in the diffuse reflection function. Namely, the diffuse reflection function $\rho(x,\mu;x',\mu')$ which depends on four independent variables is expressed by the auxiliary function $\varphi(x';x,\mu)$ depending on three independent variables (two frequency variables (x',x) and one angular variable μ). Here, the value of x' denotes the dimensionless frequency of the quantum incident on the medium, and x is the corresponding frequency of the quantum diffusely reflected from the medium. The values μ' and μ represent the cosines of the corresponding angles of incidence and diffuse reflection in relation to the boundary normal of the medium. Thus, a direct generalization of the classical solution of Ambartsumian (1)-(2) in the case of the general law $r(x'',x''')$ of redistribution of radiation by frequencies was obtained in Pikichian (1978, 1980) in the form of

$$\rho(x,\mu;x',\mu') = \frac{\lambda}{2\mu'}\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\varphi(x'';x,\mu)\,r(x'',x''')\,\varphi(x''';x',\mu')\,dx''dx'''}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}, \tag{3}$$

where a "non-coherent" analogue of Ambartsumian's auxiliary function is introduced by means of

$$\varphi(x',x;\mu) = \delta(x'-x) + \int_0^1 \rho(x,\mu;x',\mu')\,d\mu'. \tag{4}$$

From (3) and (4) followed the corresponding Ambartsumian's functional equation

$$\varphi(x',x;\mu) = \delta(x'-x) + \frac{\lambda}{2\mu'}\int_{-\infty}^{+\infty}\varphi(x'';x,\mu)\,dx''\int_{-\infty}^{+\infty} r(x'',x''')\,dx'''\int_0^1 \frac{\varphi(x''';x',\mu')}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}d\mu'. \tag{5}$$

Here value $\alpha(x)$ is the absorption profile. The ratios (3)-(5) in particular naturally turn to the above-mentioned well-known solution of the complete redistribution of radiation by frequencies, or to the results obtained by using a bilinear representation of the general law of radiation redistribution by frequency. To do this, it is enough to directly use the substitutions $r(x,x') = \alpha(x)\,\delta(x-x')$ or $r(x,x') = \sum_j A_j\alpha_j(x)\alpha_j(x')$, respectively.

Thus, to achieve the separation of variables in both anisotropic and incoherent scattering problems, traditionally used the representation of the characteristics of the elementary act of scattering through a

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bilinear series of some specially selected eigenfunctions system. In Pikichyan (2023) for the solving the same problems a new approach was proposed, where there is no need of decomposition or any special representation of the characteristics of a single act of scattering. In this paper, the advantages of the proposed approach in relation to the traditional method of decomposition of the characteristics of a single act of scattering were discussed using two examples: a one-dimensional problem of incoherent scattering and a monochromatic problem of anisotropic scattering. Analytically, the bilateral relationship between the results of both approaches was also shown. In the proposed new approach, in contrast to the well-known one, instead of the characteristics of a single act of scattering, in the form of decomposition according to a certain system of eigenfunctions, the slightly modified resulting characteristics of the radiation field (i. e., field of multiple interaction of radiation with matter) are sought. The physical basis for the expediency of such an approach is the obvious fact that during the process of multiple scattering of radiation in the medium, in each subsequent act of scattering, due to the presence of mathematical procedures of integration, the radiation field in the medium becomes more and more smooth. As a convergent series, with the same required accuracy of calculation, its description will obviously be simpler (i.e., it will include a smaller number of series members) than through a preliminary decomposition of the initial "unsmoothed" characteristics of a single act of scattering.

In problem (3)-(5), as noted above, the separation of angular variables has already been achieved naturally due to the isotropy of scattering. At the same time, it can be seen that the frequency variables in the case of the general law of radiation redistribution are not separated, since the single act of scattering itself does not possess this property.

The purpose of this work is to further simplify the solution of the problem of diffuse reflection (3)-(5) by achieving the separation of frequency variables without decomposition or any special representation of the general law - $r(x, x')$ of radiation redistribution by frequencies. This is achieved by applying the approach proposed in Pikichyan (2023).

2. Mathematical formulation of the problem

Let's introduce a new value by means of the

$$K\left(x, \mu; x', \mu'\right) \equiv \mu' \left[\frac{\alpha(x)}{\mu} + \frac{\alpha(x')}{\mu'}\right] \rho\left(x, \mu; x', \mu'\right). \tag{6}$$

It is well known that the symmetry of the elementary act of scattering also implies the symmetry of the resulting field of multiple scattering, i.e., the function of diffuse reflection (see, for example, Pikichian, 1980):

$$r\left(x'', x'''\right) = r\left(x''', x''\right) \Rightarrow \rho\left(x, \mu; x', \mu'\right) \mu' = \rho\left(x', \mu'; x, \mu\right) \mu, \tag{7}$$

but from the ratios (7) and (6), in turn, follows the symmetry of the introduced function

$$K\left(x, \mu; x', \mu'\right) = K\left(x', \mu'; x, \mu\right). \tag{8}$$

Let the kernel $K(x, \mu; x', \mu')$ will be the function you are looking for. Its knowledge unambiguously determines the solution of the problem of diffuse reflection $\rho(x, \mu; x', \mu')$.

For the integral operator $\mathrm{A} \equiv \int_{-\infty}^{+\infty} \int_0^1 K(x, \mu; x', \mu') \ldots d\mu' dx'$ let's formulate the eigenvalues $\Lambda \equiv (\nu_1, \ldots, \nu_N)$ and the eigenvector $\overrightarrow{\beta} \equiv (\beta_1(x, \mu), \ldots, \beta_N(x, \mu))$ problem in the form $\mathrm{A}\overrightarrow{\beta} = \Lambda\overrightarrow{\beta}$, with $N{=}\infty$:

$$\nu_j \beta_j(x, \mu) = \int_{-\infty}^{+\infty} \int_0^1 K\left(x, \mu; x', \mu'\right) \beta_j\left(x', \mu'\right) d\mu' dx', \tag{9}$$

see, e.g., Vasilyeva & Tikhonov (1989), there is also the condition of orthonormality

$$\int_{-\infty}^{+\infty} \int_0^1 \beta_i(x, \mu) \beta_j(x, \mu) d\mu dx = \delta_{ij}. \tag{10}$$

Continuous, positive, and symmetric kernel $K(x, \mu; x', \mu')$ let's roughly replace with a bilinear series the partial sum of eigenfunctions $\beta_j(x, \mu)$ with a finite number of terms - N :

$$K\left(x, \mu; x', \mu'\right) \sim K_N\left(x, \mu; x', \mu'\right) = \sum_{j=1}^{N} \nu_j \beta_j(x, \mu) \beta_j\left(x', \mu'\right). \tag{11}$$

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In (11), obviously, there is already a joint separation of frequency and angular variables in relation to the parameters of the quantum entering and exiting the medium, but neither the kernel itself - $K\left(x,\mu;x',\mu'\right)$, neither its proper functions - $\beta_j\left(x,\mu\right)$ nor its eigenvalues - ν_j are yet known. The challenge is to find them.

3. Analytical solution of the problem

First, we get an analogue of the Ambartsumian's functional equation for the desired kernel $K\left(x,\mu;x',\mu'\right)$. From the ratios (3)-(4) and (6) it follows:

$$\frac{2}{\lambda}K\left(x,\mu;x',\mu'\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\varphi\left(x'';x,\mu\right)r\left(x'',x'''\right)\varphi\left(x''';x',\mu'\right)dx''dx''', \tag{12}$$

$$\varphi\left(x';x,\mu\right)=\delta\left(x'-x\right)+\int_0^1\frac{K\left(x,\mu;x',\mu'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}\frac{d\mu'}{\mu'}, \tag{13}$$

and by setting (12) to (13) we arrive at the equation

$$\frac{2}{\lambda}K\left(x,\mu;x',\mu'\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\delta\left(x''-x\right)+\int_0^1\frac{K\left(x,\mu;x'',\mu''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}\right]r\left(x'',x'''\right)$$

$$\left[\delta\left(x'''-x'\right)+\int_0^1\frac{K\left(x',\mu';x''',\mu'''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x''')}{\mu'''}}\frac{d\mu'''}{\mu'''}\right]dx''dx''', \tag{14}$$

which, when the square brackets are opened, will take the form

$$\frac{2}{\lambda}K\left(x,\mu;x',\mu'\right)=r\left(x,x'\right)+\int_{-\infty}^{+\infty}r\left(x,x''\right)dx''\int_0^1\frac{K\left(x',\mu';x'',\mu''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}$$

$$+\int_{-\infty}^{+\infty}r\left(x'',x'\right)dx''\int_0^1\frac{K\left(x,\mu;x'',\mu''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}+$$

$$+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\int_0^1\frac{K\left(x,\mu;x'',\mu''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}r\left(x'',x'''\right)\int_0^1\frac{K\left(x',\mu';x''',\mu'''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x''')}{\mu'''}}\frac{d\mu'''}{\mu'''}\right]dx''dx'''. \tag{15}$$

Thus, the eigenfunctions problem (9) needs to be solved for a previously unknown kernel that satisfies the nonlinear "Ambartsumian's functional equation" (15).

By introducing an auxiliary function,

$$w(x'';x',\mu')\equiv\int_0^1\frac{K\left(x',\mu';x'',\mu''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}, \tag{16}$$

equation (15) is rewritten as

$$\frac{2}{\lambda}K\left(x,\mu;x',\mu'\right)=r\left(x,x'\right)+\int_{-\infty}^{+\infty}r\left(x,x''\right)w(x'';x',\mu')dx''$$

$$+\int_{-\infty}^{+\infty}r\left(x',x''\right)w(x'';x,\mu)dx''+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}w(x'';x,\mu)r\left(x'',x'''\right)w(x''';x',\mu')dx''dx'''. \tag{17}$$

From (17) and (9) it is not difficult to derive a system of equations for finding eigenfunctions $\beta_j\left(x,\mu\right)$

$$\frac{2}{\lambda}\nu_j\beta_j\left(x,\mu\right)=Z_j\left(x\right)+\int_{-\infty}^{+\infty}r\left(x,x''\right)\overline{w}_j\left(x''\right)dx''$$

$$+\int_{-\infty}^{+\infty}Z_j\left(x''\right)w(x'';x,\mu)dx''+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}w(x'';x,\mu)r\left(x'',x'''\right)\overline{w}_j\left(x'''\right)dx''dx''', \tag{18}$$

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where are:

$$Z_j(x) \equiv \int_{-\infty}^{+\infty} r(x,x')\,\overline{\beta}_j(x')\,dx', \quad \overline{\beta}_j(x') \equiv \int_0^1 \beta_j(x',\mu')\,d\mu', \tag{19}$$

$$\overline{w}_j(x'') \equiv \int_{-\infty}^{+\infty}\int_0^1 w(x'';x',\mu')\beta_j(x',\mu')\,d\mu'dx'. \tag{20}$$

Taking into account in the ratio (16) of the desired bilinear form (11) of the kernel $K(x,\mu;x',\mu')$, gives the value $w(x'';x',\mu')$:

$$w(x'';x',\mu') = \sum_{k=1}^{n}\nu_k w_k(x'',x',\mu'), \;\; w_k(x'',x',\mu') = \beta_k(x',\mu')\int_0^1 \frac{\beta_k(x'',\mu'')}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}. \tag{21}$$

By accounting (20)-(21), the value $\overline{w}_j(x'')$ is rewritten as:

$$\overline{w}_j(x'') = \sum_{k=1}^{N}\nu_k w_{kj}(x''), \quad w_{kj}(x'') \equiv \int_{-\infty}^{+\infty}\int_0^1 w_k(x'',x',\mu')\,\beta_j(x',\mu')\,d\mu'dx'. \tag{22}$$

Substituting (21) and (22) in (18) we finally get the expression

$$\frac{2}{\lambda}\nu_j\beta_j(x,\mu) = Z_j(x) + \sum_{k=1}^{N}\nu_k D_{kj}(x,\mu) + \sum_{k=1}^{N}\sum_{l=1}^{N}\nu_k\nu_l V_{klj}(x,\mu), \tag{23}$$

where the following symbols are entered:

$$D_{kj}(x,\mu) \equiv \int_{-\infty}^{+\infty} r(x,x'')\,w_{kj}(x'')\,dx'' + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_k(x'',x,\mu)\,r(x'',x')\,\overline{\beta}_j(x')\,dx'dx'', \tag{24}$$

$$V_{klj}(x,\mu) \equiv \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_k(x'',x,\mu)\,r(x'',x''')\,w_{lj}(x''')\,dx''dx'''. \tag{25}$$

The relation (23) is a system of nonlinear integral equations for determining eigenfunctions - $\beta_j(x',\mu')$, where the eigenvalues of ν_j are not yet known. To determine the latter, it is enough to use the operator $\int_{-\infty}^{+\infty}\int_0^1 \ldots \beta_i(x,\mu)\,d\mu dx$ act on (23), taking into account the orthonormalization condition (10), a system of nonlinear algebraic equations is obtained

$$\frac{2}{\lambda}\nu_i = b_i + \sum_{k=1}^{N}\nu_k c_{ki} + \sum_{k=1}^{N}\sum_{l=1}^{n}\nu_k\nu_l f_{kli}, \tag{26}$$

where it is indicated:

$$b_i \equiv \int_{-\infty}^{+\infty}\int_0^1 Z_i(x)\,\beta_i(x,\mu)\,d\mu dx, \quad c_{ki} \equiv \int_{-\infty}^{+\infty}\int_0^1 D_{ki}(x,\mu)\,\beta_i(x,\mu)\,d\mu dx, \tag{27}$$

$$f_{kli} \equiv \int_{-\infty}^{+\infty}\int_0^1 V_{kli}(x,\mu)\,\beta_i(x,\mu)\,d\mu dx. \tag{28}$$

Accounting in (27) -(28) ratios (19), (24) -(25), (22) finally leads them to forms

$$b_i = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overline{\beta}_i(x)\,r(x,x')\,\overline{\beta}_i(x')\,dx'dx, \tag{29}$$

$$c_{ki} = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_{ki}(x'')\,r(x'',x')\,\overline{\beta}_i(x')\,dx'dx'', \tag{30}$$

$$f_{kli} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_{ki}(x'')\,r(x'',x''')\,w_{li}(x''')\,dx''dx'''. \tag{31}$$

The system of functional nonlinear integral equations (23) taking into account (24)-(25) and (19), (21)-(22) together with the system of algebraic equations (26) taking into account (29)-(31) and (19), (22) are a closed system of equations for self-consistent finding of eigenvalues ν_k and eigenfunctions $\beta_k(x',\mu')$. After determining ν_k and $\beta_k(x',\mu')$, the solution of the initial problem, according to (6) and (11), is constructed by an explicit expression

$$\rho(x,\mu;x',\mu') = \frac{1}{\mu'}\frac{\sum_{i=1}^{N}\nu_i\beta_i(x,\mu)\,\beta_i(x',\mu')}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}. \tag{32}$$

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4. To the joint calculation of systems

For the combined calculation of systems (23) and (26), it seems easiest to first use a suitably chosen system of some orthonormalized functions as a zero approximation of $\beta_i^{(0)}(x,\mu)$, and then the values $b_i^{(0)}$, $c_{ki}^{(0)}$, $f_{kli}^{(0)}$ are calculated using (29)-(31). Then, taking $\nu_i^{(0)} \equiv b_i^{(0)}$ as the zero approximation of eigenvalues, the first approximation of eigenvalues $\nu_i^{(1)}$ is computed using the right-hand side (26). Then, from these zero approximations $\beta_i^{(0)}(x,\mu)$ and $\nu_i^{(1)}$, compute the first approximation for eigenfunctions $\beta_i^{(1)}(x,\mu)$ using the right-hand side (23). Then, again, according to (26), calculate the next (in this case, the second) approximation of eigenvalues $\nu_i^{(2)}$, and with their help, through the right side (23), obtain the values of $\beta_i^{(2)}(x,\mu)$ of the second approximation of eigenfunctions, and so on. Sequential approximations

$$\begin{bmatrix}\beta_i^{(m)}(x,\mu)\\ \nu_i^{(m)}\end{bmatrix}\to\begin{bmatrix}\nu_i^{(m+1)}\\ \beta_i^{(m)}(x,\mu)\end{bmatrix}\to\begin{bmatrix}\beta_i^{(m+1)}(x,\mu)\\ \nu_i^{(m+1)}\end{bmatrix}\to\cdots\to\begin{bmatrix}\nu_i^{(M)}\\ \beta_i^{(M-1)}(x,\mu)\end{bmatrix}\to\beta_i^{(M)}(x,\mu)$$

are constructed as specified for $m=0,\ 1,2,\dots M$. At the same time, at each next step of the calculation, the accuracy of the convergence of iterations according to some suitable criterion is evaluated until a satisfactory calculation accuracy is achieved at a certain number M.

5. Relationship between the results of the method proposed here and the traditional one

In order to compare the accuracy and efficiency of the methods proposed here and the traditionally known methods, it is very expedient to derive analytic expressions of bilateral relationship of the solutions built with their help. Let us write down the traditional solution of the problem of diffuse reflection by substituting in (3) the bilinear decomposition of the frequencies redistribution function of radiation for an elementary act of scattering $r(x,x')=\sum_{i=1}^N A_i\alpha_i(x)\alpha_i(x')$, then we get:

$$\rho(x,\mu;x',\mu')=\frac{\lambda}{2\mu'}\frac{\sum_{i=1}^N A_i\varphi_i(x,\mu)\,\varphi_i(x',\mu')}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}},\quad \varphi_i(x,\mu)\equiv\int_{-\infty}^{+\infty}\alpha_i(x')\,\varphi(x';x,\mu)\,dx', \tag{33}$$

$$\varphi_i(x,\mu)=\alpha_i(x)+\frac{\lambda}{2}\sum_{i=1}^N A_i\varphi_i(x,\mu)\int_{-\infty}^{+\infty}\alpha_i(x')\,dx'\int_0^1\frac{\varphi_i\left(x',\mu'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}\frac{d\mu'}{\mu'}. \tag{34}$$

5.1. Suppose $\varphi_i(x,\mu)$ are known, the definition of $\beta_j(x,\mu)$ and ν_j is required

$$\sum_{j=1}^N\nu_j\beta_j(x,\mu)\,\beta_j\left(x',\mu'\right)=\frac{\lambda}{2}\sum_{i=1}^N A_i\varphi_i(x,\mu)\,\varphi_i\left(x',\mu'\right). \tag{35}$$

Use the operator $\int_{-\infty}^{+\infty}\int_0^1\dots\beta_j(x',\mu')\,d\mu'dx'$ to (35), taking into account (10) we get

$$\nu_j\beta_j(x,\mu)=\frac{\lambda}{2}\sum_{i=1}^N A_i\varphi_i(x,\mu)\,q_{ij},\quad q_{ij}\equiv\int_{-\infty}^{+\infty}\int_0^1\varphi_i\left(x',\mu'\right)\beta_j\left(x',\mu'\right)d\mu'dx'. \tag{36}$$

By influencing the first relation of (36) with the same operator, we arrive at the expression

$$\nu_j=\frac{\lambda}{2}\sum_{i=1}^N A_i q_{ij}^2. \tag{37}$$

Expressions (36) and (37) define eigenfunctions $\beta_j(x,\mu)$ and eigenvalues ν_j by means of the values of pre-known auxiliary functions $\varphi_i(x,\mu)$, if q_{ij} values are available.

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To find the latter, it is enough to use the operator $\int_{-\infty}^{+\infty}\int_0^1 \ldots \varphi_k(x', \mu')\, d\mu' dx'$ to affect the first of the formulas (36), then we get:

$$\nu_j q_{kj} = \frac{\lambda}{2}\sum_{i=1}^{N} A_i a_{ik} q_{ij}, \quad a_{ik} \equiv \int_{-\infty}^{+\infty}\int_0^1 \varphi_i(x, \mu)\, \varphi_k(x, \mu)\, d\mu dx, \tag{38}$$

and with the help of (37) and the first of the ratios (38) we finally arrive at the system

$$q_{kj} = \frac{\sum_{i=1}^{N} A_i a_{ik} q_{ij}}{\sum_{i=1}^{N} A_i q_{ij}^2}. \tag{39}$$

The ratios (36) and (37) together express $\beta_j(x, \mu)$ by means of $\varphi_i(x, \mu)$

$$\beta_j(x, \mu) = \frac{\sum_{i=1}^{N} A_i \varphi_i(x, \mu)\, q_{ij}}{\sum_{i=1}^{N} A_i q_{ij}^2}, \tag{40}$$

where the values of q_{ij} are of (39). The corresponding eigenvalues of ν_j are expressed in terms of previously known A_i and q_{ij} in terms of the relation (37). Now let's get the feedback ratio.

5.2. Suppose $\beta_j(x, \mu)$, are known, and $\varphi_i(x, \mu)$ is to be found through them

From the ratios (4) and the second of (33) it follows

$$\varphi_i(x, \mu) = \int_{-\infty}^{+\infty} \alpha_i(x') \left[\delta(x' - x) + \int_0^1 \rho(x, \mu; x', \mu')\, d\mu'\right] dx', \tag{41}$$

and by opening in (41) square brackets, we get the expression

$$\varphi_i(x, \mu) = \alpha_i(x) + \int_{-\infty}^{+\infty} \alpha_i(x')\, dx' \int_0^1 \rho(x, \mu; x', \mu')\, d\mu', \tag{42}$$

then, substituting (32) into (42), we arrive at the relation

$$\varphi_i(x, \mu) = \alpha_i(x) + \sum_{j=1}^{N} \nu_j \beta_j(x, \mu)\, Q_{ij}(x, \mu), \tag{43}$$

where it is indicated

$$Q_{ij}(x, \mu) \equiv \int_{-\infty}^{+\infty} \alpha_i(x')\, dx' \int_0^1 \frac{\beta_j(x', \mu')}{\frac{\alpha(x)}{\mu} + \frac{\alpha(x')}{\mu'}} \frac{d\mu'}{\mu'}. \tag{44}$$

Thus, the relations (37), (39), (40) make it possible to proceed explicitly to the solution of the same problem (i. e, the problem of determining the values of ν_j and $\beta_j(x, \mu)$), which would be obtained by finding a solution by directly decomposing the characteristics of the initial problem of multiple scattering, through a known solution (i. e, through the known auxiliary functions of $\varphi_i(x, \mu)$). And the relations (43)-(44), on the contrary, make it possible to express the solution of the problem of diffuse reflection (i. e, the problem of determining the auxiliary functions of $\varphi_i(x, \mu)$) with the decomposed characteristics of a single act of scattering (here the values of A_i and $\alpha_i(x)$ are considered known) through the solution (i. e, through the values ν_j and $\beta_j(x, \mu)$) obtained by direct decomposition of the final characteristics of multiple scattering. With the help of these explicit expressions of bilateral relationship, it is not difficult to make a comparative analysis of the accuracy and efficiency of the above two methods of solving the diffuse reflection problem. For example, suppose the auxiliary functions $\widetilde{\varphi}_j(x, \mu)$ are obtained by some predetermined precision for a given number of terms of the expansion $\widetilde{N}$, then $\widetilde{\nu}_j$ and $\widetilde{\beta}_j(x, \mu)$ are computed according to (37), (39)-(40). The same values $\widetilde{\nu}_j$ and $\widetilde{\beta}_j(x, \mu)$ are then calculated by means of the system (23), (26) and, by comparing these results with each other, it is determined at what number $N < \widetilde{N}$ the second method provides the same accuracy. When $N = \widetilde{N}$, it is also possible to estimate (using some specially selected metric) how far the values of the quantities ($\widetilde{\nu}_j$, $\widetilde{\beta}_j(x, \mu)$) and (ν_j $\beta_j(x, \mu)$) are from each other.

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6. Conclusion

In the presented work, using the new approach previously proposed by the author (Pikichyan, 2023), the problem of diffuse reflection of radiation from a plane-parallel semi-infinite medium under isotropic scattering in the case of the general law of radiation redistribution by frequencies is solved. In the resulting solution, it is possible to separate the pair (namely, the dimensionless frequency and direction of the quantum) of independent variables when entering the medium from the same pair when exiting it. The advantage of this approach over the known ones is that this separation of the independent variables is achieved without the need to first solve an additional problem of decomposition or any special representation of the characteristics of a single act of scattering. Here, the solution of the modified initial problem itself is sought directly in a decomposed form, instead of a preliminary decomposition of the characteristics of a single act of scattering. As a result, the function to be found, which depends on four independent variables explicitly, is expressed in terms of a system of auxiliary functions that depend on only two variables. For this purpose, a problem for eigenvalues and eigenfunctions is formulated for a specially selected and previously unknown kernel. There is also a bilateral relationship between the solutions of the traditionally known and the new methods, which makes it possible to directly compare their accuracy and efficiency. The presented paper also discusses the general scheme of the organization of calculations.

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References

Ambartsumian V. A., 1941, Uchyoniye Zapiski LGU (in Russian), No. 82, Seriya Math. Nauk (Astronomiya), Vipusk 11, Trudi Astron. Observ. LGU,, 12, 64

Ambartsumian V. A., 1942, Astron. Zhour. (in Russian), (Engl. transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.159-180, CSP, 2010), 19, 30

Ambartsumian V. A., 1943a, Zhour. Theor. Exp. Phys. (in Russian), (Engl. transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.187-208, CSP, 2010), 13, 323

Ambartsumian V. A., 1943b, Dokl. AN SSSR (in Russian), (Engl. transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.181-185, CSP, 2010), 38, 257

Ambartsumian V. A., 1944a, J. Phys., 8, 65

Ambartsumian V. A., 1944b, Dokl. AN SSSR (in Russian), (Engl. transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.203-208, CSP, 2010.), 43, 106

Ambartsumian V. A., 1944c, Izvestiya AN Arm. SSR, Natural Sci. (in Russian), No.1-2, 31, (Engl.transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.209-214, CSP, 2010), 1-2, 31

Engibaryan N. B., 1971, Astrophysics, 7, 340

Engibaryan N. B., Nikogosyan A. G., 1972, Astrophysics, 8, 128

Ivanov V. V., 1973, Transfer of radiation in spectral lines

Pikichian O. V., 1978, Akademiya Nauk Armianskoy SSR Doklady, 67, 151

Pikichian O. V., 1980, Soobshcheniya Byurakanskoy Observatorii (in Russian), 52, 148

Pikichyan H. V., 2023, Communications of the Byurakan Astrophysical Observatory, 70, 143

Sobolev V. V., 1963, A treatise on radiative transfer, D. Van Nostrand

Vasilyeva A. B., Tikhonov N. A., 1989, Integralniye Uravneniya, Izd, Moskovskogo Universiteta (in Russian)

doi: https://doi.org/10.52526/25792776-23.70.2-204