Reflection of Radiation from a Plane-Parallel Half-Space in the Case of Redistribution of Radiation by Frequencies and Directions

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Abstract

The method introduced in the author's two previous works is used to solve the problem of diffuse reflection from a semi-infinite plane-parallel scattering-absorbing medium in the general case of redistribution of radiation by frequencies and directions in an elementary act of scattering. A system of functional equations was obtained to determine the unknown eigenfunctions and eigenvalues. The initial problem of the six explanatory variables is explicitly expressed in terms of these eigenfunctions that depend only on three independent variables (dimensionless frequency, zenith angle, and azimuth).

Keywords: *radiative transfer, diffuse reflection problem, Ambartsumian's nonlinear functional equation, redistribution over frequencies and directions, multidimensional eigenfunctions and eigenvalues problem*

1. Introduction and purpose of the study

In the author's last two works (Pikichyan, 2023a,b), a new approach was developed to solve the problem of diffuse reflection from a semi-infinite medium. In the first work, two particular cases were considered: a one-dimensional medium with a general law of radiation redistribution by frequencies and anisotropic monochromatic scattering in a plane-parallel semi-infinite medium. In the second work, the case of isotropic scattering with redistribution of radiation by frequencies in a semi-infinite scattering-absorbing medium with plane-parallel symmetry was analyzed. In these three cases of multiple scattering problems, the separation of the independent variables was achieved without decomposition or any special representation of the characteristics of the elementary act of scattering.

In order to solve the initial problem of multiple scatterings in a scattering-absorbing medium, it is traditionally accepted (the relevant references are given in Pikichyan, 2023a,b) to first solve some auxiliary problem of representing the characteristics of a single act of scattering (i.e., the function of redistribution of radiation by frequencies or the indicatrix of scattering) in the form of a special decomposition, where "separation of variables" would take place. Then, in the course of solving the initial problem of multiple scatterings, this skillfully introduced quality of "variable separation" is automatically transferred to the final characteristics of the radiation field, i.e., to the solution of the problem of diffuse reflection of radiation from the medium. The new approach proposed in the above-mentioned works compares favorably with the traditional one because in solving the initial problem of diffuse reflection in order to achieve the final property of "variable separation", there is no need to introduce any auxiliary problem.

In more general terms, it can be argued that the solution of the problem of diffuse reflection is traditionally analytically approximated through some degenerate kernel. If, in the cases of both incoherent and anisotropic scattering, this is traditionally achieved by preliminarily posing the solution of some auxiliary problem on the mathematical description of a unit act of scattering in the form of a degenerate kernel, then in the new approach there is no need to consider any auxiliary problem on the decomposition of the characteristics of a single act of scattering. Here it is possible to express the solution of the initial problem of multiple scattering directly through some specially constructed form of the degenerate kernel, without affecting the form of the single act of scattering. The problem of radiative transfer in the general case of redistribution of radiation by frequencies and directions was considered in Engibaryan & Nikogosyan (1972a,b), where the function of redistribution by frequencies and directions was previously presented in the form of a special decomposition.

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The purpose of this study is to apply a new approach introduced in Pikichyan (2023a,b) to solve the problem of diffuse reflection in the general case of redistribution of radiation by frequencies and directions. It will be shown below that the diffuse reflection function, which formally depends on six independent variables, is explicitly expressed in terms of some specially constructed eigenfunctions that depend on only three independent variables. There is no need for any decomposition or special representation of the redistribution function by frequencies and directions.

2. Statement and solution of the problem

In the problem of diffuse reflection of radiation from a plane-parallel semi-infinite scattering-absorbing medium in the case of the general law of radiation redistribution by frequencies and directions, the nonlinear Ambartsumian's functional equation has the form (see also Engibaryan & Nikogosyan, 1972b)

$$\frac{4\pi}{\lambda}\mu'\left(\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}\right)\rho(x,\mu;x'\mu';\varphi-\varphi')=r(x,\mu;x'-\mu';\varphi-\varphi')+$$
$$\mu'\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 r(x,\mu;x'',\mu'';\varphi-\varphi'')\,\rho(x'',\mu'';x',\mu';\varphi''-\varphi')\frac{d\mu''}{\mu''}d\varphi''dx''+$$
$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \rho(x,\mu;x'',\mu'';\varphi-\varphi''')\,r(x''',-\mu''';x',-\mu';\varphi'''-\varphi')\,d\mu'''d\varphi'''dx'''+$$
$$\mu'\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \rho(x,\mu;x''',\mu''';\varphi-\varphi''')\,r(x''',-\mu''';x'',\mu'';\varphi'''-\varphi'')\cdot$$
$$\rho(x'',\mu'';x'\mu';\varphi''-\varphi')\frac{d\mu''}{\mu''}d\varphi''dx''d\mu'''d\varphi'''dx'''. \tag{1}$$

Here: $\rho(x,\mu;x'\mu';\varphi-\varphi')$ is a diffuse reflection function in a probabilistic representation, i.e., the density of the conditional probability of quantum, which one leaving a semi-infinite medium (generally speaking after multiple scatterings) with parameters (x,μ,φ) if the latter had values (x',μ',φ') at the entrance to the medium, and $r(x,\mu;x',\mu';\varphi-\varphi')$ is the redistribution function of radiation by frequencies and directions, i. e., is a similar probability density, describing a single act of scattering. At the same time, the values x and x' are the dimensionless frequencies of the quanta leaving the medium and falling on it, respectively. The values μ, φ and μ', φ' are directions correspond to them (in relation to the external normal to the boundary of the medium), μ, μ' are cosines of zenith angles, φ, φ' are the azimuths corresponding to them. is the probability of the "survival" of the quantum in the elementary act of scattering (also called the albedo of single scattering). It should be noted that there are correlations:

$$r(x,x';\gamma)=r(x,\mu;x',\mu';\varphi-\varphi'),\quad \gamma=\vec{n}\cdot\vec{n}',\quad \vec{n}\equiv(\mu,\varphi),\quad \mu\in[0,1],\quad \varphi\in[0,2\pi], \tag{2}$$

and the expressions are also valid:

$$r(x,-\mu;x',-\mu';\varphi-\varphi')=r(x,+\mu;x',+\mu';\varphi-\varphi'),$$
$$r(x,+\mu;x',-\mu';\varphi-\varphi')=r(x',+\mu';x,-\mu;\varphi'-\varphi)),$$
$$\rho(x,\mu;x'\mu';\varphi-\varphi')\,\mu'=\rho(x'\mu';x,\mu;\varphi'-\varphi)\,\mu. \tag{3}$$

With (3), the functional equation (1) is rewritten as

$$\frac{4\pi}{\lambda}\mu'\left(\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}\right)\rho(x,\mu;x'\mu';\varphi-\varphi')=r(x,\mu;x'-\mu';\varphi-\varphi')+$$
$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 r(x,\mu;x''',\mu''';\varphi-\varphi''')\,\rho(x',\mu';x''',\mu''';\varphi'-\varphi''')\,d\mu'''d\varphi'''dx'''+$$
$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \rho(x,\mu;x'',\mu'';\varphi-\varphi'')\,r(x'',\mu'';x',\mu';\varphi''-\varphi')\,d\mu''d\varphi''dx''+$$

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$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\rho\left(x,\mu;x''',\mu''';\varphi-\varphi'''\right)r\left(x''',-\mu''';x'',\mu'';\varphi'''-\varphi''\right)\cdot$$
$$\cdot\rho\left(x',\mu';x''\mu'';\varphi'-\varphi''\ \right)d\mu''d\varphi''dx''d\mu'''d\varphi'''dx'''. \tag{4}$$

Let's enter the value

$$K\left(x,\mu;x',\mu';\varphi-\varphi'\right)\equiv\mu'\left(\frac{\alpha\left(x\right)}{\mu}+\frac{\alpha\left(x'\right)}{\mu'}\right)\rho\left(x,\mu;x'\mu';\varphi-\varphi'\right). \tag{5}$$

Obviously, $K\left(x,\mu;x',\mu';\varphi-\varphi'\right)$ is a symmetrical, positive, and continuous function

$$K\left(x,\mu;x',\mu';\varphi-\varphi'\right)=K\left(x',\mu';x,\mu;\varphi-\varphi'\right), \tag{6}$$

its knowledge uniquely determines the initial value $\rho\left(x,\mu;x'\mu';\varphi-\varphi'\right)$.
Our goal is to construct the newly introduced value $K\left(x,\mu;x',\mu';\varphi-\varphi'\right)$ and by (5) finding $\rho\left(x,\mu;x'\mu';\varphi-\varphi'\right)$. Given the notation (5), equation (4) will take the form

$$\frac{4\pi}{\lambda}K\left(x,\mu;x',\mu';\varphi-\varphi'\right)=r\left(x,\mu;x'-\mu';\varphi-\varphi'\right)+$$
$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}r\left(x,\mu;x''',\mu''';\varphi-\varphi'''\right)\frac{K\left(x',\mu';x''',\mu''';\varphi'-\varphi'''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x''')}{\mu'''}}\frac{d\mu'''}{\mu'''}d\varphi'''dx'''+$$
$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\frac{K\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}r\left(x'',\mu'';x',\mu';\varphi''-\varphi'\right)\frac{d\mu''}{\mu''}d\varphi''dx''+$$
$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\frac{K\left(x,\mu;x''',\mu''';\varphi-\varphi'''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x''')}{\mu'''}}r\left(x''',-\mu''';x'',\mu'';\varphi'''-\varphi''\right)\cdot$$
$$\frac{K\left(x',\mu';x''\mu'';\varphi'-\varphi''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}d\varphi''dx''\frac{d\mu'''}{\mu'''}d\varphi'''dx'''. \tag{7}$$

Thus, we need to build the desired kernel $K\left(x,\mu;x',\mu';\varphi-\varphi'\right)$ given by a nonlinear multidimensional functional integral equation (7). For the introduced symmetric kernel $K\left(x,\mu;x',\mu';\varphi-\varphi'\right)$ substitute the multivariate problem for the eigenvalues ν_i and the eigenfunctions $\beta_i\left(x,\mu,\varphi\right)$

$$\nu_i\beta_i\left(x,\mu,\varphi\right)=\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}K\left(x,\mu;x'\mu';\varphi-\varphi'\right)\beta_i\left(x',\mu',\varphi'\right)d\mu'd\varphi'dx', \tag{8}$$

with the orthonormalization condition

$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\beta_i\left(x,\mu,\varphi\right)\beta_j\left(x,\mu,\varphi\right)d\mu d\varphi dx=\delta_{ij}. \tag{9}$$

From (7) and (8) with regard to (9) it is not difficult to obtain the expression

$$\frac{4\pi}{\lambda}\nu_i\beta_i\left(x,\mu,\varphi\right)=Z_i\left(x,\mu,\varphi\right)+\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}r\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)Q_i\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi''dx''+$$
$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\frac{K\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}U_i\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi''dx''+$$
$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\frac{K\left(x,\mu;x''',\mu''';\varphi-\varphi'''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x''')}{\mu'''}}r\left(x''',-\mu''';x'',\mu'';\varphi'''-\varphi''\right)\cdot$$

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$$\cdot\, Q_i\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx''\frac{d\mu'''}{\mu'''}d\varphi''' dx'''. \tag{10}$$

The following designations were adopted here:

$$Z_i\left(x,\mu,\varphi\right)\equiv\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 r\left(x,\mu;x',-\mu';\varphi-\varphi'\right)\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx', \tag{11}$$

$$Q_i\left(x'',\mu'',\varphi''\right)\equiv\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \frac{K\left(x',\mu';x'',\mu'';\varphi'-\varphi''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx', \tag{12}$$

$$U_i\left(x'',\mu'',\varphi''\right)\equiv\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 r\left(x'',\mu'';x'\mu';\varphi''-\varphi'\right)\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx'. \tag{13}$$

Symmetrical, positive, and continuous kernel $K\left(x,\mu;x',\mu';\varphi-\varphi'\right)$ can be approximated by a bilinear series

$$K\left(x,\mu;x',\mu';\varphi-\varphi'\right)\sim K_N\left(x,\mu;x',\mu';\varphi-\varphi'\right)=\sum_{m=1}^N \nu_m\beta_m\left(x,\mu,\varphi\right)\beta_m\left(x',\mu',\varphi'\right). \tag{14}$$

Using the decomposition (14), the ratio (10) can be rewritten as

$$\frac{4\pi}{\lambda}\nu_i\beta_i\left(x,\mu,\varphi\right)=Z_i\left(x,\mu,\varphi\right)+\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 r\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)Q_i\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx''+$$

$$\sum_{m=1}^N \nu_m\beta_m\left(x,\mu,\varphi\right)\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \frac{\beta_m\left(x'',\mu'',\varphi''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}U_i\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx''+$$

$$\sum_{m=1}^N \nu_m\beta_m\left(x,\mu,\varphi\right)\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \frac{\beta_m\left(x''',\mu''',\varphi'''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x''')}{\mu'''}}r\left(x''',-\mu''';x'',\mu'';\varphi'''-\varphi''\right)\cdot$$

$$\cdot\, Q_i\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx''\frac{d\mu'''}{\mu'''}d\varphi''' dx'''\ . \tag{15}$$

Accounting (14) to (12) gives

$$Q_i\left(x'',\mu'',\varphi''\right)\equiv\sum_{m=1}^N \nu_m\beta_m\left(x'',\mu'',\varphi''\right)\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \frac{\beta_m\left(x',\mu',\varphi'\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx'. \tag{16}$$

Substitution (16) to (15) leads to the equation

$$\frac{4\pi}{\lambda}\nu_i\beta_i\left(x,\mu,\varphi\right)=Z_i\left(x,\mu,\varphi\right)+\sum_{m=1}^N \nu_m\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 r\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)\beta_m\left(x'',\mu'',\varphi''\right)\cdot$$

$$\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \frac{\beta_m\left(x,\mu',\varphi'\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx'\frac{d\mu''}{\mu''}d\varphi'' dx''\ +$$

$$\sum_{m=1}^N \nu_m\beta_m\left(x,\mu,\varphi\right)\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \frac{\beta_m\left(x'',\mu'',\varphi''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}U_i\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx''+$$

$$\sum_{m=1}^N\sum_{n=1}^N \nu_m\nu_n\beta_m\left(x,\mu,\varphi\right)\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1\int_{-\infty}^{+\infty}\int_0^{2\pi}\int_0^1 \frac{\beta_m\left(x''',\mu''',\varphi'''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x''')}{\mu'''}}r\left(x''',-\mu''';x'',\mu'';\varphi'''-\varphi''\right)\beta_n\left(x'',\mu'',\varphi''\right)\cdot$$

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$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\frac{\beta_n\left(x',\mu',\varphi'\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx'\frac{d\mu''}{\mu''}d\varphi'' dx''\frac{d\mu'''}{\mu'''}d\varphi''' dx'''. \tag{17}$$

Let's introduce the notations:

$$w_{ni}\left(x'',\mu'',\varphi''\right)\equiv\beta_n\left(x'',\mu'',\varphi''\right)\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\frac{\beta_n\left(x',\mu',\varphi'\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx',$$

$$D_{mi}\left(x,\mu,\varphi\right)\equiv\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}r\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)w_{mi}\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx''+\beta_m\left(x,\mu,\varphi\right)\cdot$$

$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\frac{\beta_m\left(x'',\mu'',\varphi''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}d\varphi'' dx''\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}r\left(x'',\mu'';x'\mu';\varphi''-\varphi'\right)\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx',$$

$$V_{mni}\left(x,\mu,\varphi\right)\equiv\beta_m\left(x,\mu,\varphi\right)\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\frac{\beta_m\left(x''',\mu''',\varphi'''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x''')}{\mu'''}}r\left(x''',-\mu''';x'',\mu'';\varphi'''-\varphi''\right)\cdot$$

$$\cdot\, w_{ni}\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx''\frac{d\mu'''}{\mu'''}d\varphi''' dx'''. \tag{18}$$

Then equation (17) for finding eigenfunctions $\beta_i\left(x,\mu,\varphi\right)$ will finally be written in the form

$$\frac{4\pi}{\lambda}\nu_i\beta_i\left(x,\mu,\varphi\right)=Z_i\left(x,\mu,\varphi\right)+\sum_{m=1}^{N}\nu_m D_{mi}\left(x,\mu,\varphi\right)+\sum_{m=1}^{N}\sum_{n=1}^{N}\nu_m\nu_n V_{mni}\left(x,\mu,\varphi\right). \tag{19}$$

With the help of the orthonormalization condition (9), it is not difficult to derive from the system (19) a nonlinear system of algebraic equations for finding the eigenvalues ν_i ,

$$\frac{4\pi}{\lambda}\nu_i=b_i+\sum_{m=1}^{N}\nu_m c_{mi}+\sum_{m=1}^{N}\sum_{n=1}^{N}\nu_m\nu_n f_{mni}\ , \tag{20}$$

where the notations are entered:

$$b_i\equiv\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}Z_i\left(x,\mu,\varphi\right)\beta_i\left(x,\mu,\varphi\right)d\mu d\varphi dx, \tag{21}$$

$$c_{mi}\equiv\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}D_{mi}\left(x,\mu,\varphi\right)\beta_i\left(x,\mu,\varphi\right)d\mu d\varphi dx, \tag{22}$$

$$f_{mni}\equiv\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}V_{mni}\left(x,\mu,\varphi\right)\beta_i\left(x,\mu,\varphi\right)d\mu d\varphi dx. \tag{23}$$

Taking into account the ratios (11) and (8) in the notations (21)-(23) gives them the form:

$$b_i\equiv\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\beta_i\left(x,\mu,\varphi\right)r\left(x,\mu;x',-\mu';\varphi-\varphi'\right)\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx' d\mu d\varphi dx \tag{24}$$

$$c_{mi}\equiv\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\Big[\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}r\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)w_{mi}\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx''+$$

$$\beta_m\left(x,\mu,\varphi\right)\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\frac{\beta_m\left(x'',\mu'',\varphi''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}d\varphi'' dx''\cdot$$

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$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1} r\left(x'',\mu'';x'\mu';\varphi''-\varphi'\right)\beta_i\left(x',\mu',\varphi'\right)d\mu' d\varphi' dx'\right]\beta_i\left(x,\mu,\varphi\right)d\mu d\varphi dx\ , \tag{25}$$

$$f_{mni}\equiv$$
$$\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1} w_{mi}\left(x''',\mu''',\varphi'''\right)r\left(x''',-\mu''';x'',\mu'';\varphi'''-\varphi''\right)w_{ni}\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx''\frac{d\mu'''}{\mu'''}d\varphi''' dx'''. \tag{26}$$

By opening the square brackets and taking into account the first notation of (18), the ratio (25) is given a simpler shape

$$c_{mi}\equiv 2\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\int_{-\infty}^{+\infty}\int_{0}^{2\pi}\int_{0}^{1}\beta_i\left(x,\mu,\varphi\right)r\left(x,\mu;x'',\mu'';\varphi-\varphi''\right)w_{mi}\left(x'',\mu'',\varphi''\right)\frac{d\mu''}{\mu''}d\varphi'' dx'' d\mu dx d\varphi. \tag{27}$$

Thus, it follows from (5) and (14) that the explicit solution of the problem of diffuse reflection in the general case of redistribution of radiation in frequencies and directions is given by the expression

$$\rho\left(x,\mu;x'\mu';\varphi-\varphi'\right)=\frac{\sum_{m=1}^{N}\nu_m\beta_m\left(x,\mu,\varphi\right)\beta_m\left(x',\mu',\varphi'\right)}{\mu'\left(\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}\right)}\ , \tag{28}$$

where the values of $\beta_m\left(x,\mu,\varphi\right)$ satisfy the system of nonlinear functional integral equations (19) taking into account the notations (18), and the values ν_m satisfy the system of nonlinear algebraic equations (20) taking into account the notations (24), (26), (27).

The general scheme of the organization of the joint calculation of systems (19) and (20) does not differ from those described in the two previous works (Pikichyan 2023a, b). There is also no difference in the general line of calculations for obtaining an analytical two-way relationship between the solutions of the traditional and developed approaches, so we will not dwell on them here, in order to avoid unnecessary cumbersomeness.

3. Conclusion

In this paper, using the approach introduced in the author's two previous works (Pikichyan 2023a, b), conventionally called the method of "decomposition of the resultant field (DRF)", it is shown that the desired diffuse reflection function $\rho\left(x,\mu;x'\mu';\varphi-\varphi'\right)$ formally dependent on six independent quantities, is expressed in terms of some specially constructed eigenfunctions $\beta_m\left(x,\mu,\varphi\right)$ dependent on only three variables. To compute the desired eigenfunctions and their corresponding eigenvalues, a system consisting of nonlinear functional integral and nonlinear algebraic equations was obtained. The physical basis for the effectiveness of the developed approach lies in the fact that in the process of multiple scattering of the quantum, with each subsequent act of scattering, the field becomes more and more "smooth", due to a certain mathematical procedure of its integration with the characteristics of previous scatterings. As a result, it is easier to represent the "smoothed" characteristics of the multiple scattering field by decomposition into a series according to some specially constructed self-consistent system of orthonormalized functions than to use the same procedure for the characteristics of the primary "unsmoothed" field of a single scattering act.

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