Complete simplification of the solution of the diffuse reflection problem by the method of "decomposition of resultant field" (DRF)

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Abstract

The classical diffuse reflection problem (DRP) of radiation from a plane-parallel semi-infinite medium under isotropic scattering in the case of the general law $r(x, x')$ of radiation redistribution by frequencies in the elementary act of scattering is considered. It is shown that the solution of the DRP $\rho\left(x, \mu; x^{'}, \mu^{'}\right)$, which depends on four independent variables, with the help of author's recently developed method of "decomposition of the resultant field-DRF" it is possible to express explicitly through auxiliary functions $G_i(z)$, $Q_i(x)$ that depend on only one independent variable. A significant difference between this DRF method and the widely used traditional approach of "decomposition of a single act - DSA" of scattering is that it no longer requires any decomposition or a special representation of the characteristics of a single act of scattering (in the case under consideration, the function of redistribution of radiation by frequencies - $r(x, x')$) . In DRF, the resulting field of multiply scattered radiation itself is searched directly in the decomposed form. It is obvious that after each successive single act of scattering, the resulting radiation field in the medium becomes more and more smoother, so that with the same required accuracy, its decomposition will contain fewer terms than the decomposition of the characteristic of a single act of scattering. This circumstance makes the DRF method especially effective in relation to the DSA approach in cases where the latter requires taking into account a large number of terms of the decomposition of the frequency redistribution function $r(x, x')$. Explicit analytical expressions of the two-way relationship are also obtained between the desired values of the traditional and the methods proposed by the author.

Keywords: *radiative transfer, diffuse reflection problem, redistribution of radiation by frequencies, Ambartsumian's nonlinear functional equation, eigenfunctions in radiative transfer, decomposition of the resulting field*

1. Introduction and purpose of the work

The classical diffuse reflection problem (DRP) of radiation from a scattering-absorbing semi-infinite plane-parallel medium in astrophysics has a key importance in the analysis and interpretation of stellar and planetary spectra, as well as in the problems of radiative energy transfer in media where the processes of multiple interactions of radiation with matter take place. The solution of DRP depends on many independent variables, so in theoretical astrophysics, precise and approximate methods have traditionally been developed, to reduce it to the definition of auxiliary functions that depend on a smaller number of variables (Chandrasekhar, 1950, Sobolev, 1956, 1972). In the well-known methods of solving the DRP, the achievement of the separation of variables of the resulting field of multiply scattered radiation is still traditionally achieved with the help of preliminary application of variable separation procedures in the description of the single scattering act itself (below, we will combine such methods, under the conditional name of the approach "decomposition of a single act (DSA)" of scattering). As a result, the variables are also automatically separated in the desired solution of the DRP, i.e. in the resulting field of multiply scattered radiation (see, for example, Ambartsumian's pioneering work in the case of anisotropic scattering (Ambartsumian, 1943, 1944) . However, in those applications where the solution DRP requires taking into account a large number of terms (for example, in (Smokity & Anikonov, 2008) it is about 150 and more) in the decomposition of

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doi: https://doi.org/10.52526/25792776-24.71.2-275

the characteristics of a single act of scattering (for example, scattering indicatrix in anisotropic scattering or frequency redistribution functions in noncoherent scattering), this approach obviously becomes less effective. Indeed: a) for the characteristics of the elementary act of scattering, the construction of an optimal decomposition with the required accuracy is already a separate difficult task, and b) in the case of a large number of terms in the single act expansion, the number of nonlinear integral equations in the final system of solving the DRP also increases, which significantly complicates the calculation of the resulting field of multiply scattered radiation. Therefore, there is a need to search for a new, simpler, and more effective way to solve the DRP, free from these difficulties.

In the author's latest works (Pikichyan, 2023a,b,c, 2024), a method of so-called decomposition of the resulting field (DRF) of radiation was developed to solve the DRP. The DRF approach compares favorably with the hitherto widely used traditional REA method in that the separation of independent variables in the solution of the problem is achieved without any decomposition or special representation of the characteristic of the elementary act of scattering. Here the description of the final field of multiple scattering is looking for in a decomposed form and there is no need for any preliminary decomposition or a special representation of the characteristics of a single act of scattering. The effectiveness of the DRF method is based on the obvious fact that with each successive act of scattering, the resulting field of diffuse radiation becomes more and more smoother. As a result, the representation, with a certain accuracy, of this field of multiply scattered radiation in the form of a bilinear series should naturally include a smaller number of terms of decomposition than the representation, with the same accuracy, of the characteristics of a single act of scattering. With this approach: a) there is no need to preliminarily consider the non-simple problem of decomposition of characteristics of the single act of scattering and b) the number of equations and auxiliary functions in the final system of nonlinear integral equations in the DRF approach should be significantly less than in the traditional DSA method. In (Pikichyan, 2023a) , using the DRF method, the case of anisotropic scattering was considered, where each azimuthal harmonic of the DRP solution dependent on two angular variables was explicitly expressed in terms of specially constructed eigenfunctions depending on only one angular variable. In the same work (Pikichyan, 2023a), in a similar way the separation of frequency variables x, and x' was achieved in the case of one-dimensional isotropic scattering medium with redistribution of radiation over frequencies under the general law $r\left(x,x'\right)$. Further, in (Pikichyan, 2023b), a three-dimensional case of DRP in the case of noncoherent isotropic scattering with the same redistribution function $r\left(x,x'\right)$ was considered, as a result, the solution $\rho\left(x,\mu;x',\mu'\right)$ which depended on four variables: two frequency variables $\left(x,x'\right)$ and two angular variables $\left(\mu,\mu'\right)$, using DRF were explicitly expressed in terms of eigenfunctions $\beta_i\left(x,\mu\right)$ dependent on only two independent variables. A more general case of DRP, when in a single act of scattering take place redistribution of radiation both: as in frequencies as of directions, was considered in (Pikichyan, 2023c). Here the solution of DRP $\rho\left(x,\mu;x',\mu';\varphi{-}\varphi'\right)$ was expressed through the specially introduced eigenfunctions $\beta_i\left(x,\mu,\varphi\right)$. And finally, in the report (Pikichyan, 2024), the main results of the work (Pikichyan, 2023b) were presented in a slightly more compact form without mathematical calculations. In the works (Pikichyan, 2023a,b, 2024), explicit analytical relations of the two-way relationship between the auxiliary functions of the DRF and DSA methods were also given, and possible general schemes for the organization of numerical calculations were described.

In this paper, the analysis of the DRP, studied in the works (Pikichyan, 2023b, 2024), is continued with the aim of further, even greater simplification of its solution. In these works (Pikichyan, 2023b, 2024) it has already been shown that the desired value $\rho\left(x,\mu:x'.\mu'\right)$ depends on four independent variables is expressed in terms of some specially constructed eigenfunctions $\beta_i\left(x,\mu\right)$ that depend on only two independent variables. No decomposition procedure or special representation of the frequency redistribution function was used. It is shown below that the solution of the DRP in the case of incoherent isotropic scattering under an arbitrary redistribution law of radiation by frequencies can be simplified even further. Eigenfunctions $\beta_i\left(x,\mu\right)$ that depend on two independent variables can be reduced to the construction of two functions $G_i\left(z\right)$ and $Q_i\left(x\right)$, which have only one independent variable each. Here, $z=\frac{\mu}{\alpha(x)}$ is the combined variable, and the value of $\alpha\left(x\right)$, as usual, represents the absorption coefficient profile in the elementary act of scattering. The solution of the DRP is the density of the conditional probability $\rho\left(x,\mu:x',\mu'\right)$ of the diffuse reflection of a quantum from a semiinfinite medium: at a dimensionless frequency x, in the direction of μ (cosine of the angle of incidence of the quantum in relation to the normal to the boundary of the medium) in a solid angle of $2\pi d\mu$, provided that it enters the medium at a frequency x' , from the direction μ'.

doi: https://doi.org/10.52526/25792776-24.71.2-275

2. Initial ratios

In the case of isotropic scattering, the solution of the DRP is given by the expression (Pikichian, 1978, 1980)

$$\rho\left(x,\mu;x',\mu'\right)=\frac{\lambda}{2\mu'}\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\varphi\left(x'';x,\mu\right)r\left(x'',x'''\right)\varphi\left(x''';x',\mu'\right)dx''dx'''}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}} \tag{1}$$

where the auxiliary function is given by:

$$\varphi\left(x',x;\mu\right)=\delta\left(x'-x\right)+\int_{0}^{1}\rho\left(x,\mu;x',\mu'\right)d\mu' \tag{2}$$

$$\varphi\left(x',x;\mu\right)=\delta\left(x'-x\right)+\frac{\lambda}{2}\int_{-\infty}^{+\infty}\varphi\left(x'';x,\mu\right)dx''\int_{-\infty}^{+\infty}r\left(x'',x'''\right)dx'''\int_{0}^{1}\frac{\varphi\left(x''';x',\mu'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}\frac{d\mu'}{\mu'} \tag{3}$$

In (1) the natural separation of angular variables takes place due to the isotropy of scattering, i.e. the desired function $\rho\left(x,\mu;x',\mu'\right)$ of the four independent variables is naturally expressed through the auxiliary function $\varphi\left(x',x;\mu\right)$ which has the smaller number, the three, of variables. In the works (Pikichyan, 2023b, 2024), a new function was introduced through the desired function $\rho\left(x,\mu;x',\mu'\right)$

$$K\left(x,\mu;x',\mu'\right)\equiv\mu'\left[\frac{\alpha\left(x\right)}{\mu}+\frac{\alpha\left(x'\right)}{\mu'}\right]\rho\left(x,\mu;x',\mu'\right) \tag{4}$$

for which, using the DRF method, a bilinear expansion was constructed through its eigenfunctions $\beta_j\left(x,\mu\right)$ in the n-th approximation,

$$K\left(x,\mu;x',\mu'\right)\sim K_n\left(x,\mu;x',\mu'\right)=\sum_{j=1}^{n}\nu_j\beta_j\left(x,\mu\right)\beta_j\left(x',\mu'\right)\ . \tag{5}$$

From expressions (2), (4) and (5) it follows

$$\varphi\left(x',x;\mu\right)=\delta\left(x'-x\right)+\sum_{j=1}^{n}\nu_j\beta_j\left(x,\mu\right)\int_{0}^{1}\frac{\beta_j\left(x',\mu'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}\frac{d\mu'}{\mu'} \tag{6}$$

where the eigenfunctions $\beta_j\left(x,\mu\right)$ satisfy a system of nonlinear integral equations

$$\frac{2}{\lambda}\nu_j\beta_j\left(x,\mu\right)=Z_j\left(x;[\beta]\right)+\sum_{k=1}^{n}\nu_kD_{kj}\left(x,\mu;[\beta^3]\right)+\sum_{k=1}^{n}\sum_{l=1}^{n}\nu_k\nu_lV_{klj}\left(x,\mu;[\beta^5]\right)\ . \tag{7}$$

Here the designations were introduced:

$$Z_j\left(x;[\beta]\right)\equiv\int_{-\infty}^{+\infty}r\left(x,x'\right)\overline{\beta}_j\left(x'\right)dx'\ ,\qquad \overline{\beta}_j\left(x'\right)\equiv\int_{0}^{1}\beta_j\left(x',\mu'\right)d\mu' \tag{8}$$

$$D_{kj}\left(x,\mu;[\beta^3]\right)\equiv\int_{-\infty}^{+\infty}r\left(x,x''\right)w_{kj}\left(x''\right)dx''+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}w_k\left(x'',x,\mu\right)r\left(x'',x'\right)\overline{\beta}_j\left(x'\right)dx'dx'', \tag{9}$$

$$V_{klj}\left(x,\mu;[\beta^5]\right)\equiv\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}w_k\left(x'',x,\mu\right)r\left(x'',x'''\right)w_{lj}\left(x'''\right)dx''dx'''\ , \tag{10}$$

$$w_{kj}\left(x''\right)\equiv\int_{-\infty}^{+\infty}\int_{0}^{1}w_k\left(x'',x',\mu'\right)\beta_j\left(x',\mu'\right)d\mu'dx', \tag{11}$$

$$w_k\left(x'',x',\mu'\right)=\beta_k\left(x',\mu'\right)\int_{0}^{1}\frac{\beta_k\left(x'',\mu''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}\ . \tag{12}$$

The eigenvalues ν_i satisfy a nonlinear algebraic system

doi: https://doi.org/10.52526/25792776-24.71.2-275

$$\frac{2}{\lambda}\nu_i = b_i + \sum_{k=1}^{n} \nu_k c_{ki} + \sum_{k=1}^{n}\sum_{l=1}^{n} \nu_k \nu_l f_{kli} \ , \tag{13}$$

The designations are accepted here:

$$b_i = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \overline{\beta}_i\left(x\right) r\left(x,x'\right) \overline{\beta}_i\left(x'\right) dx'dx \ , \quad c_{ki} = 2\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} w_{ki}\left(x''\right) r\left(x'',x'\right) \overline{\beta}_i\left(x'\right) dx'dx'', \tag{14}$$

$$f_{kli} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_{ki}\left(x''\right) r\left(x'',x'''\right) w_{li}\left(x'''\right) dx''dx''' \tag{15}$$

3. Separation of variables in the desired eigenfunctions $\beta_j\left(x,\mu\right)$

It is easy to see that the integral in (12) depends only on the variable x'' and the combination of $\frac{\alpha(x')}{\mu'}$, of parameters x', and μ' of the quantum entering the medium. This property makes it possible in formulas (7)-(12) to switch to a new variable $\frac{\alpha(x)}{\mu} = \frac{1}{z}$, denoting $\widetilde{\beta}_j\left(x,z\right) = \beta_j\left(x,\mu\right)$, instead of (7) we get

$$\frac{2}{\lambda}\nu_j \widetilde{\beta}_j\left(x,z\right) = Q_j\left(x\right) + \sum_{k=1}^{n} \nu_k \widetilde{\beta}_k\left(x,z\right) \widetilde{q}_{kj}\left(z\right) \ , \tag{16}$$

from which the ratios immediately follow:

$$\frac{2}{\lambda}\nu_j G_{ij}\left(z\right) = \delta_{ij} + \sum_{k=1}^{n} \nu_k G_{ik}\left(z\right) \widetilde{q}_{kj}\left(z\right) \ , \tag{17}$$

$$\widetilde{\beta}_j\left(x,z\right) = \sum_{i=1}^{n} Q_i\left(x\right) G_{ij}\left(z\right) \ . \tag{18}$$

Thus, the definition of eigenfunctions $\widetilde{\beta}_j\left(x,z\right)$, which depend on two independent variables, can be reduced to finding two auxiliary functions - $Q_i\left(x\right)$ and $G_{ij}\left(z\right)$, each of which depends on only one independent variable. From the ratios (1), (6) and (18) it follows that the solution of the initial DRP can be analytically simplified in three stages:

$$\rho\left(x,\mu;x',\mu'\right) \longrightarrow \varphi\left(x',x;\mu\right) \longrightarrow \widetilde{\beta}_k\left(x,z\right) \longrightarrow Q_i\left(x\right) \ , \ G_{ij}\left(z\right). \tag{19}$$

4. Orthonormalization conditions

An eigenfunctions of the kernel $K\left(x,\mu;x',\mu'\right)$ appearing in (4) and (5) were introduced in (Pikichyan, 2023b, 2024) by means of equations

$$\nu_j \beta_j\left(x,\mu\right) = \int_{-\infty}^{+\infty}\int_0^1 K\left(x,\mu;x',\mu'\right) \beta_j\left(x',\mu'\right) d\mu' dx' \tag{20}$$

and meet the orthonormalization condition

$$\int_{-\infty}^{+\infty}\int_0^1 \beta_i\left(x,\mu\right) \beta_j\left(x,\mu\right) d\mu dx = \delta_{ij} \ . \tag{21}$$

It is not difficult to show (see Appendix A) that condition (21) for auxiliary functions $Q_m\left(x\right)$ and $G_{mj}\left(z\right)$ will be rewritten in two equivalent forms:

$$\sum_{m=1}^{n}\sum_{l=1}^{n} \int_{-\infty}^{+\infty} \alpha\left(x\right) Q_m\left(x\right) Q_l\left(x\right) \mathcal{G}_{mlij}\left(x\right) dx = \delta_{ij} \ , \tag{22}$$

$$\sum_{m=1}^{n}\sum_{l=1}^{n} \int_0^{\infty} G_{mi}\left(z\right) L_{ml}\left(z\right) G_{lj}\left(z\right) dz = \delta_{ij} \ , \tag{23}$$

doi: https://doi.org/10.52526/25792776-24.71.2-275

where the values $\mathcal{G}_{mlij}(x)$ and $L_{ml}(z)$ are introduced by means of notation:

$$\mathcal{G}_{mlij}(x) \equiv \int_0^{\frac{1}{\alpha(x)}} G_{mi}(z)\, G_{lj}(z)\, dz \ , \tag{24}$$

$$L_{mh}(z') \equiv 2 \int_{E(z')} \alpha(x')\, Q_m(x')\, Q_h(x')\, dx' \ . \tag{25}$$

5. Finding auxiliary functions $Q_j(x)$ and $G_{ik}(z'')$ of a one variable

5.1. Functional equation for $Q_j(x)$

For the auxiliary function $Q_j(x)$ a functional equation is obtained (see Appendix B)

$$Q_j(x) = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} r(x, x'')\, Q_i(x'')\, S_{ij}(x'', [Q^2])\, dx'' \ , \tag{26}$$

in which the value of $S_{ij}(x'', [Q^2])$ is given by the expression

$$S_{ij}(x'', [Q^2]) \equiv p_{ij}(x'')\, \alpha(x'') + \sum_{k=1}^{n} \nu_k \sum_{l=1}^{n} \sum_{m=1}^{n} \int_{-\infty}^{+\infty} H_{ijkml}(x'', x')\, Q_m(x')\, Q_l(x')\, \alpha(x')\, dx' \ . \tag{27}$$

Taking into account the condition of orthonormalization (22) gives expression (27) a form,

$$S_{ij}(x'', [Q^2]) = p_{ij}(x'') \left[\alpha(x'') + \nu_j\right] - \sum_{k=1}^{n} \nu_k \sum_{l=1}^{n} \sum_{m=1}^{n} \int_{-\infty}^{+\infty} \widetilde{H}_{ikmlj}(x'', x')\, Q_m(x')\, Q_l(x')\, \alpha(x')\, dx', \tag{28}$$

in which it is very important (see "b" of section 6.2), that j-th eigenvalue of ν_j is out of the sign of sum. In (27) - (28) the following designations are also adopted:

$$p_{ij}(x') \equiv \int_0^{\frac{1}{\alpha(x')}} G_{ij}(z')dz' \tag{29}$$

$$H_{ijkml}(x'', x') \equiv \int_0^{\frac{1}{\alpha(x'')}} G_{ik}(z'')\, dz'' \int_0^{\frac{1}{\alpha(x')}} \frac{G_{mk}(z')\, G_{lj}(z')}{z' + z''} z' dz' \ , \tag{30}$$

$$\widetilde{H}_{ijkml}(x'', x') \equiv \int_0^{\frac{1}{\alpha(x'')}} G_{ik}(z'')\, z'' dz'' \int_0^{\frac{1}{\alpha(x')}} \frac{G_{mk}(z')\, G_{lj}(z')}{z'' + z'} dz' \ . \tag{31}$$

From the comparison of (29) - (31) taking into account (23) follows the relationship

$$H_{ijkml}(x'', x') = p_{ij}(x'')\, \mathcal{G}_{mlkj}(x') - \widetilde{H}_{ijkml}(x'', x') \ . \tag{32}$$

5.2. Functional equation for $G_{ik}(z)$

It can be shown that the value $\tilde{q}_{kj}(z)$, which appears in (16)-(17), by means of the relation (7)-(12) (see Appendix C) takes the form

$$\begin{gathered}\widetilde{q}_{kj}(z) \equiv \sum_{i=1}^{n} z \int_0^{\infty} \frac{G_{ik}(z'')}{z + z''} dz'' \cdot \\ \cdot \sum_{l=1}^{n} \left\{ \int_0^{\infty} M_{il}(z'', z')\, G_{lj}(z')\, dz' + \nu_l \sum_{s=1}^{n} \int_0^{\infty} F_{is}(z'', z''')\, G_{sl}(z''')\, \mathcal{L}_{lj}(z'''; [G^2])\, dz''' \right\}\end{gathered} \tag{33}$$

where the permutation was used

doi: https://doi.org/10.52526/25792776-24.71.2-275

$$\int_{-\infty}^{+\infty}\dots dx\int_{0}^{\frac{1}{\alpha(x)}}\dots dz=2\int_{0}^{\infty}\dots dz\int_{E(z)}\dots dx\ ,\qquad E\left(z\right)=\left\{x:\alpha\left(x\right)\leq\frac{1}{z}\right\} \tag{34}$$

and following designations are adopted:

$$\mathcal{L}_{lj}\left(z''';\left[G^{2}\right]\right)\equiv\int_{0}^{\infty}\frac{\sum_{m=1}^{n}\sum_{h=1}^{n}G_{ml}\left(z'\right)L_{mh}\left(z'\right)G_{hj}\left(z'\right)}{z'''+z'}z'dz'\ , \tag{35}$$

$$M_{il}\left(z'',z'\right)\equiv4\int_{E(z'')}\int_{E(z')}Q_{i}\left(x''\right)r\left(x'',x'\right)Q_{l}\left(x'\right)\alpha\left(x'\right)dx'dx''\ , \tag{36}$$

$$F_{is}\left(z'',z'''\right)\equiv4\int_{E(z'')}\int_{E(z''')}Q_{i}\left(x''\right)r\left(x'',x'''\right)Q_{s}\left(x'''\right)dx'''dx''\ . \tag{37}$$

From the relations (33) and (17) the functional equation for the quantities $G_{ik}\left(z''\right)$ is obtained

$$\frac{2}{\lambda}\nu_{j}G_{pj}\left(z\right)=\delta_{pj}+z\sum_{k=1}^{n}\nu_{k}G_{pk}\left(z\right)\sum_{i=1}^{n}\int_{0}^{\infty}\frac{G_{ik}\left(z''\right)}{z+z''}W_{ij}\left(z'',\left[G^{3}\right]\right)dz''\ , \tag{38}$$

where the designations are accepted

$$W_{ij}\left(z'',\left[G^{3}\right]\right)\equiv\sum_{l=1}^{n}\left[\int_{0}^{\infty}M_{il}\left(z'',z'\right)G_{lj}\left(z'\right)dz'+\nu_{l}\sum_{s=1}^{n}\int_{0}^{\infty}F_{is}\left(z'',z'''\right)G_{sl}\left(z'''\right)\mathcal{L}_{lj}\left(z''';\left[G^{2}\right]\right)dz'''\right]. \tag{39}$$

The use of orthonormalization (23) gives expression (39) form

$$\begin{gathered}W_{ij}\left(z'',\left[G^{3}\right]\right)=\sum_{s=1}^{n}\int_{0}^{\infty}M_{is}\left(z'',z'\right)G_{sj}\left(z'\right)dz'+\nu_{j}\sum_{k=1}^{n}\int_{0}^{\infty}F_{ik}\left(z'',z'''\right)G_{kj}\left(z'''\right)dz'''-\\-\sum_{l=1}^{n}\nu_{l}\sum_{s=1}^{n}\int_{0}^{\infty}F_{is}\left(z'',z'''\right)G_{sl}\left(z'''\right)\widetilde{\mathcal{L}_{lj}}\left(z''';\left[G^{2}\right]\right)z'''dz'''\end{gathered} \tag{40}$$

where the value is entered

$$\widetilde{\mathcal{L}_{lj}}\left(z''';\left[G^{2}\right]\right)\equiv\int_{0}^{\infty}\frac{\sum_{m=1}^{n}\sum_{h=1}^{n}G_{ml}\left(z'\right)L_{mh}\left(z'\right)G_{hj}\left(z'\right)}{z'''+z'}dz'\ , \tag{41}$$

which satisfies of the expression

$$\mathcal{L}_{lj}\left(z''';\left[G^{2}\right]\right)=\delta_{lj}-z'''\,\widetilde{\mathcal{L}_{lj}}\left(z''';\left[G^{2}\right]\right). \tag{42}$$

6. Algebraic systems for finding the eigenvalues

In the functional equations (26)-(27) and (38)-(39), the eigenvalues of problem (20) are still to be determined

6.1. A system of nonlinear algebraic equations

To determine these eigenvalues, there is already a system of nonlinear algebraic equations (13), and it remains only to express the quantities included in it through auxiliary functions of one variable $Q_i\left(x\right)$ and $G_{ij}\left(z\right)$. After simple but somewhat cumbersome calculations (see Appendix D). for the quantities b_i, c_{kj}, f_{kli} appear in the algebraic system, the following expressions are obtained:

$$b_{i}=\sum_{k=1}^{n}\sum_{j=1}^{n}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Omega_{ki}\left(x\right)r\left(x,x'\right)\Omega_{ji}\left(x'\right)dx'dx=\sum_{k=1}^{n}\sum_{j=1}^{n}\int_{0}^{\infty}\int_{0}^{\infty}G_{ki}\left(z\right)U_{kj}\left(z,z'\right)G_{ji}\left(z'\right)dz'dz\ , \tag{43}$$

doi: https://doi.org/10.52526/25792776-24.71.2-275

$$c_{kj} = 2\sum_{s=1}^{n}\sum_{l=1}^{n}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Gamma_{skj}\left(x''\right)Q_s\left(x''\right)r\left(x'',x'\right)\Omega_{lj}\left(x'\right)dx'dx'' , \tag{44}$$

$$f_{kli} = \sum_{s=1}^{n}\sum_{h=1}^{n}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Gamma_{ski}\left(x''\right)Q_s\left(x''\right)r\left(x'',x'''\right)Q_h\left(x'''\right)\Gamma_{hli}\left(x'''\right)dx'''dx'' , \tag{45}$$

where auxiliary quantities have the form:

$$\Omega_{ki}\left(x\right) \equiv Q_k\left(x\right)\alpha\left(x\right)p_{ki}\left(x\right) , \tag{46}$$

$$\Gamma_{skj}\left(x''\right) = \sum_{q=1}^{n}\sum_{m=1}^{n}\int_{-\infty}^{+\infty}H_{sjkqm}\left(x''x'\right)Q_q\left(x'\right)Q_m\left(x'\right)\alpha\left(x'\right)dx' = \int_{0}^{\frac{1}{\alpha(x'')}}G_{sk}\left(z''\right)\mathcal{L}_{kj}\left(z''';\left[G^2\right]\right)dz''. \tag{47}$$

6.2. A system of linear algebraic equations for determining eigenvalues

In addition to the system of **nonlinear** algebraic equations (13), a system of **linear** algebraic equations can also be obtained to determine the eigenvalues of ν_j (see Appendix E).

$$\nu_j = \mathcal{A}_j + \sum_{k=1}^{n}\nu_k\mathcal{B}_{ij} , \tag{48}$$

where the values $\mathcal{A}_j$ and $\mathcal{B}_{ij}$ are given by expressions:

$$\mathcal{A}_j \equiv \frac{\int_{-\infty}^{+\infty}Q_j\left(x\right)dx - \sum_{i=1}^{n}\int_{-\infty}^{+\infty}\alpha\left(x''\right)\Omega_{ij}\left(x''\right)dx''}{\sum_{i=1}^{n}\int_{-\infty}^{+\infty}\Omega_{ij}\left(x''\right)dx''} , \tag{49}$$

$$\mathcal{B}_{kj} \equiv \frac{\sum_{i=1}^{n}\sum_{l=1}^{n}\sum_{m=1}^{n}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\alpha\left(x''\right)Q_i\left(x''\right)\widetilde{H}_{ikmlj}\left(x'',x'\right)Q_m\left(x'\right)Q_l\left(x'\right)\alpha\left(x'\right)dx'dx''}{\sum_{i=1}^{n}\int_{-\infty}^{+\infty}\Omega_{ij}\left(x''\right)dx''} \tag{50}$$

7. The final system of equations for solving the DRP

The above equations (26), (38), (13), (48) together define a final system consisting of nonlinear integral equations and two types of algebraic equations (nonlinear and linear) to find the auxiliary functions $Q_j\left(x\right)$, $G_{ji}\left(z\right)$ and the eigenvalues ν_j.

$$\begin{cases} \frac{2}{\lambda}\nu_j G_{pj}\left(z\right) = \delta_{pj} + z\sum_{k=1}^{n}\nu_k G_{pk}\left(z\right)\sum_{i=1}^{n}\int_0^{\infty}\frac{G_{ik}(z'')}{z+z''}W_{ij}\left(z'',\left[G^3\right]\right)dz'' \\ Q_j\left(x\right) = \sum_{i=1}^{n}\int_{-\infty}^{+\infty}r\left(x,x''\right)Q_i\left(x''\right)S_{ij}\left(x'',\left[Q^2\right]\right)dx'' \\ \frac{2}{\lambda}\nu_i = b_i + \sum_{k=1}^{n}\nu_k c_{ki} + \sum_{k=1}^{n}\sum_{l=1}^{n}\nu_k\nu_l f_{kli} \quad or \quad \nu_j = \mathcal{A}_j + \sum_{k=1}^{n}\nu_k\mathcal{B}_{ij} \end{cases} \tag{51}$$

8. The final forms of explicit solution of the DRP, which were obtained by two methods, DSA and DRF

Now it is not difficult to give an explicit form of the solution of the DRP obtained by both methods: the new DRF and the traditional DSA.

8.1. Explicit form of solution using the DRF method

In terms of the auxiliary functions of a single variable $Q_i\left(x\right)$ and $G_{ij}\left(z\right)$ taking into account the notation $\rho\left(x,\mu;x',\mu'\right) \equiv \widetilde{\rho}\left(x,z;x',z'\right)$ from (4), (5) and (18) we get the expression

$$\widetilde{\rho}\left(x,z;x',z'\right) = \frac{1}{\alpha\left(x'\right)}\sum_{i=1}^{n}\sum_{k=1}^{n}Q_i\left(x\right)R_{ik}\left(z,z'\right)Q_k\left(x'\right) , \tag{52}$$

doi: https://doi.org/10.52526/25792776-24.71.2-275

where the value of $R_{ik}(z,z')$ is given as

$$R_{ik}(z,z') = z\frac{\sum_{j=1}^{n}\nu_j G_{ij}(z)\, G_{kj}(z')}{z+z'} \ . \tag{53}$$

From (52) - (53) it can be seen that when solving the DRP using the DRF method, similar to the traditional DSA method, the classical structure of separation of independent variables is preserved (compare, below, with formula (58) - (59).

8.2. An explicit solution of DRP using the traditional DSA method

To compare the DRF and DSA methods, it is also appropriate to cite here the solution of DRP using the traditional DSA method (see and compare with the works (Engibaryan & Nikogosyan, 1972a,b). From the relation (1) by substituting the bilinear decomposition of the function of redistribution of radiation by frequencies in a single scattering act

$$r(x'',x''') = \sum_{i=1}^{N} A_i\alpha_i(x'')\,\alpha_i(x''') \ , \tag{54}$$

a well-known solution is easily obtained

$$\widetilde{\rho}(x,z;x',z') = \frac{\lambda}{2}\frac{z}{\alpha(x')}\frac{\sum_{i=1}^{N} A_i\widetilde{\varphi}_i(x,z)\,\widetilde{\varphi}_i(x',z')}{z+z'} \ , \tag{55}$$

where auxiliary function $\widetilde{\varphi}_i(x,z)$ satisfies the analogue of Ambartsumian's functional equation

$$\widetilde{\varphi}_i(x,z) = \alpha_i(x) + \frac{\lambda}{2}z\sum_{j=1}^{n} A_j\widetilde{\varphi}_j(x,z)\int_{-\infty}^{+\infty}\alpha_i(x')\,dx'\int_0^{\frac{1}{\alpha(x')}}\frac{\widetilde{\varphi}_j(x',z')}{z+z'}dz' \ . \tag{56}$$

It is also easy to show that the auxiliary function $\widetilde{\varphi}_i(x,z)$ is representable in the form (see Appendix F).

$$\widetilde{\varphi}_i(x,z) = \sum_{k=1}^{N}\alpha_k(x)\,g_{ki}(z) \ , \tag{57}$$

then the explicit DRP decision obtained by the DSA method will take the form of

$$\widetilde{\rho}(x,z;x',z') = \frac{1}{\alpha(x')}\sum_{k=1}^{N}\sum_{j=1}^{N}\alpha_k(x)\,\mathcal{R}_{kj}(z,z')\,\alpha_j(x') \ , \tag{58}$$

where the value of $\mathcal{R}_{kj}(z,z')$ is given as

$$\mathcal{R}_{kj}(z,z') \equiv \frac{\lambda}{2}z\frac{\sum_{i=1}^{N} A_i g_{ki}(z)\, g_{ji}(z')}{z+z'} \ . \tag{59}$$

The auxiliary function of one argument $g_{jk}(z)$, which appears in (59), satisfies the functional equation

$$g_{jk}(z) = \delta_{jk} + \frac{\lambda}{2}z\sum_{i=1}^{N} A_i g_{ji}(z)\int_0^{\infty}\frac{\sum_{h=1}^{N} u_{hk}(z')\, g_{hi}(z')}{z+z'}dz' \ , \tag{60}$$

where the value of $u_{jk}(z')$ is given by the expression

$$u_{jk}(z') \equiv 2\int_{E(z)}\alpha_j(x)\,\alpha_k(x)\,dx \ . \tag{61}$$

9. Two-way relationship between the auxiliary functions of the DRF and DSA techniques

For a practical assessment of the relative effectiveness of the DRF and DSA methods, it is very useful to provide a two-way analytical relationship between the functions that determine the solution of the DRP.

doi: https://doi.org/10.52526/25792776-24.71.2-275

9.1. The $\widetilde{\varphi}_k(x',z')$ are known, i.e. the values: A_i, $\alpha_s(x)$ and $g_{si}(z)$ it is necessary to find $\widetilde{\beta}_j(x,z)$ and ν_j.

Explicit expressions are obtained (see Appendix G):

$$\widetilde{\beta}_j(x,z) = \frac{\sum_{k=1}^{n} A_k q_{kj} \sum_{s=1}^{n} \alpha_s(x) g_{sk}(z)}{\sum_{k=1}^{N} A_k q_{kj}^2}\ , \qquad \nu_j = \frac{\lambda}{2}\sum_{i=1}^{n} A_i q_{ij}^2\ , \tag{62}$$

where the values q_{kj} are determined from the algebraic system

$$q_{ij} = \frac{\sum_{k=1}^{n} A_k a_{ik} q_{kj}}{\sum_{k=1}^{N} A_k q_{kj}^2}\ . \tag{63}$$

The a_{ik} coefficients in expression (63) are as follows

$$a_{ik} = \sum_{s=1}^{n}\sum_{l=1}^{n}\int_{-\infty}^{+\infty} \alpha(x)\,\alpha_s(x)\,\Theta_{sikl}(x)\,\alpha_l(x)\,dx = \sum_{s=1}^{n}\sum_{l=1}^{n}\int_{0}^{\infty} g_{si}(z)\,\Lambda_{sl}(z)\,g_{lk}(z)\,dz\ , \tag{64}$$

where the symbols are used:

$$\Theta_{sikl}(x) \equiv \int_0^{\frac{1}{\alpha(x)}} g_{si}(z) g_{lk}(z)\,dz\ , \qquad \Lambda_{sl}(z) \equiv 2\int_{E(z)} \alpha(x)\,\alpha_s(x)\,\alpha_l(x)\,dx\ . \tag{65}$$

9.2. Known $\widetilde{\beta}_j(x,z)$, i.e. the values $Q_i(x)$ and $G_{ij}(z)$, it is necessary to find $\widetilde{\varphi}_i(x,z)$

From the relation (57) it can be seen that the auxiliary function $\widetilde{\varphi}_i(x,z)$, which depends on two independent variables (x,z), is explicitly expressed in terms of two functions, $\alpha_s(x)$ and $g_{ki}(z)$, by only one variable. At the same time, the functions $\alpha_s(x)$ in advance are already known from the construction of the expansion (54), so in this DRP only the functions of $g_{si}(z)$ are sought. To define them, an explicit expression is obtained (see Appendix H).

$$g_{ki}(z) = \delta_{ki} + z\sum_{j=1}^{n} \nu_j \tilde{G}_{kj}(z) \sum_{h=1}^{n}\int_0^{\infty} \frac{G_{hj}(z')}{z+z'}\gamma_{hi}(z')\,dz'\ , \tag{66}$$

where the designations are accepted:

$$\tilde{G}_{kj}(z) \equiv \sum_{m=1}^{n} \vartheta_{km} G_{mj}(z)\ , \qquad \vartheta_{km} \equiv \int_{-\infty}^{+\infty} \vartheta(x)\,\alpha_k(x)\,Q_m(x)\,dx\ , \tag{67}$$

$$\gamma_{hi}(z') \equiv 2\int_{E(z')} Q_h(x')\,\alpha_i(x')\,dx', \tag{68}$$

the value $\vartheta(x)$ is a given weight function that appears in the orthonormalization condition

$$\int_{-\infty}^{+\infty} \vartheta(x)\,\alpha_k(x)\,\alpha_s(x)\,dx = \delta_{ks}.$$

.

10. Conclusion

It was shown above that the proposed DRF method, in contrast to the traditional DSA approach, makes it possible in the case of isotropic scattering under the general law of radiation redistribution by frequencies to achieve complete simplification of the solution of the DRP from a semi-infinite plane-parallel medium, without any decomposition or special representation of the characteristics of a single act of scattering, i.e., of the radiation redistribution function by frequencies $r(x'',x''')$. At the same time, the solution of the DRP, which depends on four independent variables, as in the traditional DSA method, can be reduced to the definition of auxiliary functions of only one independent variable. Explicit analytical expressions of the two-way relationship between the desired functions of the traditional and the methods proposed by the author are also obtained.

doi: https://doi.org/10.52526/25792776-24.71.2-275

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doi: https://doi.org/10.52526/25792776-24.71.2-275

Appendices

Appendix A

In relation (21), we replace the variable $\frac{\alpha(x)}{\mu} = \frac{1}{z}$, taking into account the notation $\widetilde{\beta}_j(x,z) = \beta_j(x,\mu)$ we get

$$\int_{-\infty}^{+\infty} \alpha(x)\, dx \int_0^{\frac{1}{\alpha(x)}} \widetilde{\beta}_i(x,z)\, \widetilde{\beta}_j(x,z)\, dzdx = \delta_{ij}\ . \tag{A.1}$$

From (18) we have the expressions

$$\widetilde{\beta}_i(x,z) = \sum_{m=1}^{n} Q_m(x)\, G_{mi}(z)\ , \qquad \widetilde{\beta}_j(x,z) = \sum_{l=1}^{n} Q_l(x)\, G_{lj}(z)\ , \tag{A.2}$$

the substitution of which in (A.1) gives the relation

$$\sum_{m=1}^{n}\sum_{l=1}^{n} \int_{-\infty}^{+\infty} \alpha(x)\, Q_m(x)\, Q_l(x)\, dx \int_0^{\frac{1}{\alpha(x)}} G_{mi}(z)\, G_{lj}(z)\, dzdx = \delta_{ij}\ . \tag{A.3}$$

Let's change the order of integration in the expression (A.3) following the example

$$\int_{-\infty}^{+\infty} \dots dx \int_0^{\frac{1}{\alpha(x)}} \dots dz = 2\int_0^{\infty} \dots dz \int_{E(z)} \dots dx\ , \tag{A.4}$$

then the ratio (A.3) will take the form of

$$\sum_{m=1}^{n}\sum_{l=1}^{n} \int_0^{\infty} G_{mi}(z)\, G_{lj}(z)\, dz \cdot 2\int_{E(z)} \alpha(x)\, Q_m(x)\, Q_l(x)\, dx = \delta_{ij}\ . \tag{A.5}$$

Taking into account in expressions (A.3) and (A.5) the notations:

$$\mathcal{G}_{mlij}(x) \equiv \int_0^{\frac{1}{\alpha(x)}} G_{mi}(z)\, G_{lj}(z)\, dz, \qquad L_{mh}(z) \equiv 2\int_{E(z)} \alpha(x)\, Q_m(x)\, Q_h(x)\, dx \tag{A.6}$$

leads to two forms of orthonormalization conditions in the form of relations (22) and (23).

Appendix B

From formulas (7) - (12) it is not difficult to obtain

$$Q_j(x) = Z_j(x) + \sum_{k=1}^{n} \nu_k \int_{-\infty}^{+\infty} r(x,x'')\, dx'' \int_{-\infty}^{+\infty} \int_0^{\frac{1}{\alpha(x')}} \widetilde{\beta}_k(x',z')\, \tilde{B}_k(x'',z')\, \widetilde{\beta}_j(x',z')\, dz'dx', \tag{B.1}$$

where the ratio was used

$$\int_0^1 \frac{\beta_k(x'',\mu'')}{\frac{\alpha(x')}{\mu'} + \frac{\alpha(x'')}{\mu''}} \frac{d\mu''}{\mu''} \equiv B_k\left(x'', \frac{\alpha(x')}{\mu'}\right) \equiv \widetilde{B}_k(x'',z') = \sum_{i=1}^{n} Q_i(x'')z' \int_0^{\frac{1}{\alpha(x'')}} \frac{G_{ik}(z'')}{z'+z''} dz'', \tag{B.2}$$

In the last integral (B.2), the expression (18) is also taken into account. For the value $Z_j(x)$ appearing in (B.1) taking into account (18), we get:

$$Z_j(x) \equiv \int\limits_{-\infty}^{+\infty} r(x,x')\, dx' \int\limits_0^1 \beta_j(x',\mu')d\mu' = \int\limits_{-\infty}^{+\infty} r(x,x')\, \alpha(x')\, dx' \int\limits_0^{\frac{1}{\alpha(x')}} \widetilde{\beta}_j(x',z')dz' =$$

doi: https://doi.org/10.52526/25792776-24.71.2-275

$$= \int\limits_{-\infty}^{+\infty} r\left(x,x'\right) dx' \sum_{i=1}^{n} Q_i\left(x'\right) \alpha\left(x'\right) \int\limits_{0}^{\frac{1}{\alpha(x')}} G_{ij}\left(z'\right) dz'$$

$$Z_j\left(x\right) = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} r\left(x,x''\right) Q_i\left(x''\right) p_{ij}\left(x''\right) \alpha\left(x''\right) dx'' \ , \tag{B.3}$$

where indicated

$$p_{ij}\left(x'\right) \equiv \int_{0}^{\frac{1}{\alpha(x')}} G_{ij}\left(z'\right) dz' \ . \tag{B.4}$$

Taking into account (B.2) and (B.3) from (B.1), we obtain the expression

$$Q_j\left(x\right) = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} r\left(x,x''\right) Q_i\left(x''\right) p_{ij}\left(x''\right) \alpha\left(x''\right) dx'' +$$

$$+ \sum_{i=1}^{n} \int_{-\infty}^{+\infty} r\left(x,x''\right) Q_i\left(x''\right) dx'' \sum_{k=1}^{n} \nu_k \sum_{m=1}^{n} \sum_{l=1}^{n} \int_{-\infty}^{+\infty} H_{ijkml}\left(x'',x'\right) Q_m\left(x'\right) Q_l\left(x'\right) \alpha\left(x'\right) dx' \ , \tag{B.5}$$

from which follow (26)-(31).

Appendix C

With the help of relations (7)-(12) it is not difficult to arrive at the expression

$$\frac{2}{\lambda} \nu_j \beta_j\left(x,\mu\right) = Q_j\left(x\right) + \sum_{k=1}^{n} \nu_k \beta_k\left(x,\mu\right) q_{kj}\left(\frac{\alpha\left(x\right)}{\mu}\right) \ , \tag{C.1}$$

where the value of $q_{kj}\left(\frac{\alpha(x)}{\mu}\right)$ has the form

$$q_{kj}\left(\frac{\alpha\left(x\right)}{\mu}\right) \equiv \int_{-\infty}^{+\infty} B_k\left(x'',\frac{\alpha\left(x\right)}{\mu}\right) dx'' \int_{-\infty}^{+\infty} r\left(x'',x'\right) \overline{\beta}_j\left(x'\right) dx' +$$

$$+ \sum_{l=1}^{n} \nu_l \int_{-\infty}^{+\infty} B_k\left(x'',\frac{\alpha\left(x\right)}{\mu}\right) dx'' \int_{-\infty}^{+\infty} r\left(x'',x'''\right) w_{lj}\left(x'''\right) dx''' \ , \tag{C.2}$$

in this case, the value of $B_k\left(x'',\frac{\alpha(x)}{\mu}\right)$ is given by the expression (B.2). By switching to the variable z and taking into account $q_{kj}\left(\frac{\alpha(x)}{\mu}\right) \equiv \widetilde{q}_{kj}\left(z\right)$, we get

$$\widetilde{q}_{kj}\left(z\right) \equiv \int_{-\infty}^{+\infty} \widetilde{B}_k\left(x'',z\right) dx'' \int_{-\infty}^{+\infty} r\left(x'',x'\right) \alpha\left(x'\right) dx' \int_{0}^{\frac{1}{\alpha(x')}} \widetilde{\beta}_j\left(x',z'\right) dz' + \sum_{l=1}^{n} \nu_l \cdot$$

$$\cdot \int_{-\infty}^{+\infty} \widetilde{B}_k\left(x'',z\right) dx'' \int_{-\infty}^{+\infty} r\left(x'',x'''\right) dx''' \int_{-\infty}^{+\infty} \alpha\left(x'\right) dx' \int_{0}^{\frac{1}{\alpha(x')}} \widetilde{\beta}_l\left(x',z'\right) \widetilde{B}_l\left(x''',z'\right) \widetilde{\beta}_j\left(x',z'\right) dz' dx' \ . \tag{C.3}$$

Setting in (C.3) of the relation (18) taking into account (B.2) we come to the expression

$$\widetilde{q}_{kj}\left(z\right) = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} Q_i\left(x''\right) dx'' z \int_{0}^{\frac{1}{\alpha(x'')}} \frac{G_{ik}\left(z''\right)}{z+z''} dz'' \sum_{l=1}^{n} \int_{-\infty}^{+\infty} r\left(x'',x'\right) Q_l\left(x'\right) \alpha\left(x'\right) dx' \int_{0}^{\frac{1}{\alpha(x')}} G_{lj}\left(z'\right) dz' +$$

$$+ \sum_{l=1}^{n} \nu_l \sum_{i=1}^{n} \int_{-\infty}^{+\infty} Q_i\left(x''\right) dx'' z \int_{0}^{\frac{1}{\alpha(x'')}} \frac{G_{ik}\left(z''\right)}{z+z''} dz'' \sum_{s=1}^{n} \int_{-\infty}^{+\infty} r\left(x'',x'''\right) Q_s\left(x'''\right) dx''' \cdot$$

doi: https://doi.org/10.52526/25792776-24.71.2-275

$$\cdot \sum_{m=1}^{n} \sum_{h=1}^{n} \int_{-\infty}^{+\infty} \alpha\left(x'\right) Q_m\left(x'\right) Q_h\left(x'\right) dx' \int_0^{\frac{1}{\alpha(x')}} G_{ml}\left(z'\right) G_{hj}\left(z'\right) z' dz' \int_0^{\frac{1}{\alpha(x''')}} \frac{G_{sl}\left(z'''\right)}{z'+z'''} dz''', \tag{C.4}$$

from which it already follows (33).

Appendix D

For the output of expressions (43)-(47), the initial formulas are (13)-(15). From the latter, on the basis of expression (18), the left part of the expression (43) is first derived, then the substitution of places in the double integral is performed according to the sample (A.4) and the right part of the expression (43) is obtained. Forms (44) - (45) are also obtained in the same way, while the main quantities included in (14) - (15) are given by the expressions:

$$\overline{\beta}_i\left(x\right) = \sum_{k=1}^{n} Q_k\left(x\right) \alpha\left(x\right) p_{ki}\left(x\right) , \quad w_{ki}\left(x''\right) = \sum_{s=1}^{n} Q_s\left(x''\right) \Gamma_{ski}\left(x''\right) . \tag{D.1}$$

Appendix E

From the relations (26) and (28) by integration along x within (-∞,+∞), the relations (48)-(50) are obtained, taking into account the known relationship of the absorption profile with the frequencies redistribution function

$$\int_{-\infty}^{+\infty} r\left(x, x''\right) dx = \alpha\left(x''\right) . \tag{E.1}$$

Appendix F

To make the transition from expression (56) to form (57), note that the double integral in the right-hand side of (56) depends only on the variable z and let's take into account the notation, then

$$h_j\left(x', z\right) \equiv \int_0^{\frac{1}{\alpha(x')}} \frac{\widetilde{\varphi}_j\left(x', z'\right)}{z+z'} dz' , \quad \omega_{ji}\left(z\right) \equiv \int_{-\infty}^{+\infty} h_j\left(x', z\right) \alpha_i\left(x'\right) dx' , \tag{F.1}$$

relation (56) will take the form of a linear algebraic system

$$\widetilde{\varphi}_i\left(x, z\right) = \alpha_i\left(x\right) + \frac{\lambda}{2} z \sum_{j=1}^{N} A_j \widetilde{\varphi}_j\left(x, z\right) \omega_{ji}\left(z\right) . \tag{F.2}$$

If we now introduce the resolvent by means of the equation

$$g_{ki}\left(z\right) = \delta_{ki} + \frac{\lambda}{2} z \sum_{j=1}^{N} A_j g_{kj}\left(z\right) \omega_{ji}\left(z\right) , \tag{F.3}$$

we arrive at the explicit expression (57). By substituting (57) in (55) we immediately get (58) - (59). If we substitute the second of the relations (F.1) into (F.3), then, taking into account (57) and replacing the integrals in the double integral (according to the example (A.4)), the relation (F.1) turns into the functional equation (60).

Appendix G

In the same approximation $N = n$, we equalize the values of the function $K\left(x, \mu; x', \mu'\right)$ obtained by two methods, DRF and DSA

$$\sum_{j=1}^{n} \nu_j \beta_j\left(x, \mu\right) \beta_j\left(x', \mu'\right) = \frac{\lambda}{2} \sum_{k=1}^{n} A_k \varphi_k\left(x, \mu\right) \varphi_k\left(x', \mu'\right) . \tag{G.1}$$

doi: https://doi.org/10.52526/25792776-24.71.2-275

Taking into account the orthonormalization condition (21), the following expressions are obtained from (G.1):

$$\nu_j\beta_j\left(x,\mu\right) = \frac{\lambda}{2}\sum_{k=1}^{N} A_k\varphi_k\left(x,\mu\right)q_{kj}\ , \qquad q_{kj} \equiv \int_{-\infty}^{+\infty}\int_0^1 \varphi_k\left(x',\mu'\right)\beta_j\left(x',\mu'\right)d\mu' dx'. \tag{G.2}$$

If condition (21) is applied to the first relation of (G.2) and also applied the operator $\int\limits_{-\infty}^{+\infty}\int\limits_0^1 \varphi_k\left(x',\mu'\right)\ldots d\mu' dx'$, then taking into account the second of the conditions (G.2) it is not difficult to come to the following expressions (see (Pikichyan, 2023b, 2024)):

$$\nu_j = \frac{\lambda}{2}\sum_{k=1}^{n} A_k q_{kj}^2\ , \qquad \nu_j q_{ij} = \frac{\lambda}{2}\sum_{k=1}^{n} A_k a_{ik} q_{kj}\ , \qquad a_{ik} \equiv \int_{-\infty}^{+\infty}\int_0^1 \varphi_i\left(x',\mu'\right)\varphi_k\left(x',\mu'\right)d\mu' dx', \tag{G.3}$$

$$\beta_j\left(x,\mu\right) = \frac{\sum_{k=1}^{n} A_k\varphi_k\left(x,\mu\right)q_{kj}}{\sum_{k=1}^{n} A_k q_{kj}^2}\ , \qquad q_{ij} = \frac{\sum_{k=1}^{n} A_k a_{ik} q_{kj}}{\sum_{k=1}^{n} A_k q_{kj}^2}\ . \tag{G.4}$$

Relations (62), (64)-(65) are easily obtained from expressions (G.2)-(G.4) in the standard transition to the variable z, taking into account (57), as well as by replacing the integrals of the variables x and z in the double integral (G.3).

Appendix H

To obtain the relation (66), first substitute the expansion (54) into the expression (1), then it is not difficult to see that

$$\varphi_i\left(x,\mu\right) \equiv \int_{-\infty}^{+\infty}\varphi\left(x';x,\mu\right)\alpha_i\left(x'\right)dx'. \tag{H.1}$$

By substituting the expression

$$\varphi\left(x',x;\mu\right) = \delta\left(x'-x\right) + \int_0^1 \rho\left(x,\mu;x',\mu'\right)d\mu'\ , \tag{H.2}$$

given in (2), we get

$$\varphi_i\left(x,\mu\right) = \alpha_i\left(x\right) + \int_{-\infty}^{+\infty}\int_0^1 \rho\left(x,\mu;x',\mu'\right)\alpha_i\left(x'\right)d\mu' dx'\ . \tag{H.3}$$

Taking into account expressions (4)-(5) turns (H.3) into a ratio (see Pikichyan (2023b, 2024))

$$\varphi_i\left(x,\mu\right) = \alpha_i\left(x\right) + \sum_{k=1}^{n}\nu_k\beta_k\left(x,\mu\right)\int_{-\infty}^{+\infty}\int_0^1 \frac{\beta_k\left(x',\mu'\right)\alpha_i\left(x'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}\frac{d\mu'}{\mu'}dx'\ . \tag{H.4}$$

Proceeding in a standard way here: moving from the variable μ to the z variable, then using explicit expressions (18) and (57) taking into account the orthonormalization condition $\int_{-\infty}^{+\infty}\vartheta\left(x\right)\alpha_k\left(x\right)\alpha_s\left(x\right)dx = \delta_{ks}$ of values $\alpha_i\left(x\right)$ with the weight $\vartheta\left(x\right)$, swapping the integrals, it is not difficult to obtain the relations (66)-(68).

doi: https://doi.org/10.52526/25792776-24.71.2-275