A deformation of Master-Space and inertia effects within the theory of Master Space-Teleparallel Supergravity

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Abstract

In the framework of the theory of *Master space*-Teleparallel Supergravity ($\widetilde{MS}_p$-TSG) (Ter-Kazarian, 2024b), having the gauge *translation* group in tangent bundle, in present article we address the theory of a general deformation of the flat MS_p induced by external force exerted on a particle, subject to certain rules. Our idea is that the *universality* of gravitation and inertia attribute to the single mechanism of origin from geometry but having a different nature. We have ascribed, therefore, the inertia effects to the geometry itself but as having a nature other than 4D Riemannian space. We consider a general smooth deformation map $\Omega(\varrho)\,:\,\underline{M}_2 \to \mathcal{M}_2$ in terms of the *world - deformation* tensor Ω, the flat MS_p, and a general smooth differential 2D-manifold, $\mathcal{M}_2$. The Ω is a function of *local rate*, $\varrho(\underline{x})$, of instantaneously change of the velocity of massive test particle under the unbalanced external net force. A general deformation is composed of the two subsequent deformations $\overset{\circ}{\Omega}$: $\underline{M}_2 \to \underline{V}_2$ and $\breve{\Omega}\,:\,\underline{V}_2 \to \mathcal{M}_2$, where $\underline{V}_2$ is the 2D semi-Riemannian space, $\overset{\circ}{\Omega}$ and $\breve{\Omega}$ are the corresponding world deformation tensors. In the simple case of $\Omega = \overset{\circ}{\Omega}$, $\breve{\Omega}^{\mu}{}_{\nu} \equiv \delta^{\mu}_{\nu}$, we have to write the rate, ϱ, in terms of the Lorentz spinors $(\underline{\theta}, \underline{\bar{\theta}})$ referred to $\underline{M}_2$, and period of superoscillations (τ). The latter can be defined as a function of proper time $(\underline{s})$ induced by the *world-deformation* tensor $\Omega(\underline{s})$. In this way we show that the occurrence of the, so-called, *absolute* and *inertial* accelerations, and that the *inertial* force as well, are clearly caused by this. Therewith the *relative* acceleration in 4D Minkowski space, M_4, (both magnitude and direction, in Newton's terminology), to the contrary, has nothing to do with a deformation of $\underline{M}_2$ and, thus, it cannot produce the inertia effects. We calculate the relativistic inertial force in Minkowski, semi-Riemannian and post Riemannian spaces. This furnishes a justification for the introduction of the relativistic Weak Principle of Equivalence (WPE). We discuss the inertia effects beyond the hypothesis of locality with special emphasis on deformation $\underline{M}_2 \longrightarrow \underline{V}_2^{(\varrho)}$, which essentially improves the standard framework. Whereas we derive the tetrad fields as a function of ϱ, describing corresponding *fictitious graviton*. The *fictitious gravitino* will be arisen under infinitesimal transformations of local supersymmetry.

Keywords: *Teleparallel Supergravity–Spacetime Deformation–Inertia Effects*

1. Introduction

Using Palatini's formalism extended in a plausible fashion to the $\widetilde{MS}_p$-Supergravity (Ter-Kazarian, 2023c, 2024c), in a recent papers (Ter-Kazarian, 2024b) we reinterpret a flat $\widetilde{MS}_p$-SG theory with Weitzenböck torsion as the quantum field theory of $\widetilde{MS}_p$-TSG, having the gauge *translation* group in tangent bundle. For a benefit of the reader as a guiding principle to make the rest of paper understandable, we necessarily recount succinctly some of the highlights behind of $\widetilde{MS}_p$-SG and $\widetilde{MS}_p$-TSG in the Appendix.

A quantum field theory of $\widetilde{MS}_p$-Supergravity is a *local* extension of the theory of *global Master space* (MS_p)-SUSY (Ter-Kazarian, 2023a, 2024a). The latter, in turn, is the microscopic theory of: 1) standard Lorentz code of motion (SLC), 2) deformed Lorentz symmetry and 3) deformed geometry induced by foamy effects at the Planck scale, and tested in ultra-high energy experiments. Therewith we derive the SLC in a new perspective of global double MS_p-SUSY transformations. The MS_p-SUSY provides valuable theoretical clue for a complete revision of our standard ideas about the Lorentz code of motion to be now referred to as the *intrinsic* property of a particle. This is a result of the first importance for a really comprehensive theory of inertia. The MS_p-SUSY theory, among other things, actually explores the first part of the phenomenon of

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inertia, which refers to *inertial uniform motion along rectilinear timelike world lines.* This developments are in many ways exciting, yet mysteries remain, and some of deeper issues are still unresolved, such as those which relate the inertial effects. This comprises a second half of phenomenon of inertia, which stood one of the major unattained goals since the time traced back to the works developed by Galileo and Newton. The *principle of inertia* they developed is one of the fundamental principles of classical mechanics. This governs the *uniform motion* of a body and describes how it is affected by applied forces. Ever since, there is an ongoing quest to understand the reason for the universality of the gravitation and inertia, attributing to the WPE, which establishes the independence of free-fall trajectories of the internal composition and structure of bodies. However, the nature of the relationship of gravity and inertia continues to elude us and, beyond the WPE, there has been little progress in discovering their true relation. Viewed from the perspective of GR theory, the fictitious forces are attributed to geodesic motion in spacetime. Physicists have gone a long way in developing this theory. But nothing is reliable and such efforts do not make sense. Indeed, Einstein later showed that the field equations of GR imply that a test particle in an empty universe has inertial properties. Bondi (1952), Einstein (1917), O'Raifeartaigh et al. (2017), Sciama (1953). Thus as emphasized by Einstein, GR is failed to account for the inertial properties of matter, so that an adequate theory of inertia is still lacking.

We reinterpret the flat $\widetilde{MS}_p$-SG theory with Weitzenböck torsion as the theory of $\widetilde{MS}_p$-TSG having the gauge *translation* group in tangent bundle. An important property of Teleparallel Gravity is that its spin connection is related only to the inertial properties of the frame, not to gravitation. Whereas the Hilbert action vanishes and the gravitino action loses its spin connections, so we find that the accelerated reference frame has Weitzenböck torsion induced by gravitinos. The action of $\widetilde{MS}_p$-TSG is invariant under local translations, under local super symmetry transformations and by construction is invariant under local Lorentz rotations and under diffeomorphisms. So that this action is invariant under the Poincaré supergroup and under diffeomorphisms. The Weitzenböck connection defines the acceleration through force equation, with torsion (or contortion) playing the role of force. Thus, the results obtained clearly show that the frames expressing linear and rotational acceleration can be interpreted via torsion as an invariant property of spacetime.

In the present article, our idea is that the *universality* of gravitation and inertia attribute to the single mechanism of origin from geometry but having a different nature. We have ascribed, therefore, the inertia effects to the geometry itself but as having a nature other than 4D Riemannian space (for earlier version see (Ter-Kazarian, 2012)). We show that in the $\widetilde{MS}_p$-TSG theory the occurrence of the *absolute* and *inertial* accelerations, and the *inertial* force are obviously caused by a general deformation of the flat MS_p. While the *relative* acceleration (both magnitude and direction, in Newton's terminology) in 4D Minkowski space, M_4 , to the contrary, has nothing to do with a deformation of $\underline{M}_2$ and, thus, it cannot produce the inertia effects. We calculate the relativistic inertial force in Minkowski, semi-Riemannian and post Riemannian spaces. Despite of totally different and independent sources of gravitation and inertia, this establishes the independence of free-fall trajectories of the mass, internal composition and structure of bodies. This furnishes a justification for the introduction of the WPE. We discuss the inertia effects by going beyond the hypothesis of locality with special emphasis on deformation $\underline{M}_2 \longrightarrow \underline{V}_2^{(\varrho)}$, which essentially improves the standard framework.

With this perspective in sight, we will proceed according to the following structure. To start with, in Section 2 we briefly review a general deformation of the flat MS_p. Section 3 is devoted to the model building in background M_4. In Section 4 we discuss the inertia effects beyond the hypothesis of locality with special emphasis on deformation $\underline{M}_2 \longrightarrow \underline{V}_2^{(\varrho)}$, which essentially improves the standard framework. Whereas we derive the tetrad fields describing *fictitious graviton*. In Section 5 we calculate the inertial force in the semi-Riemannian space V_4. In Section 6 we discuss the inertial effects in the background post Riemannian geometry. We bring the concluding remarks in section 7. In Appendix, we will briefly review the theories of $\widetilde{MS}_p$-SG and $\widetilde{MS}_p$-TSG. For brevity, whenever possible undotted and dotted spinor indices often can be ruthlessly suppressed without ambiguity. Unless indicated otherwise, the natural units, $h = c = 1$ are used throughout.

2. A general deformation of the flat MS_p

In this section we will briefly discuss a general deformation of the flat MS_p induced by external force exerted on a particle, to show that in the $\widetilde{MS}_p$-SG theory the occurrence of the so-called *absolute* and

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inertial accelerations, as well as *inertial effects* (fictitious gravity) are clearly caused by this. For brevity reason, we shall forbear here to review the mathematical aspects of the spacetime deformation technique. We invite the interested reader to consult (Ter-Kazarian, 2011, 2012) for a more rigorous formulation with various applications. We now extend, for the self-contained arguments, just necessary geometrical ideas of this framework without going into the subtleties, as applied to the 2D deformation $\underline{M}_2 \to \mathcal{M}_2$. In the framework of spacetime deformation theory (Ter-Kazarian, 2011), we consider a smooth deformation map

$$\Omega(\varrho) : \underline{M}_2 \to \mathcal{M}_2, \tag{1}$$

written in terms of the *world - deformation* tensor Ω, the flat MS_p, and a general smooth differential 2D-manifold, $\mathcal{M}_2$. The *world-deformation* tensor, $\Omega(\varrho)$, is a function of *local rate*, $\varrho(\underline{x})$, of instantaneously change of the velocity ($\underline{v}^{(\pm)}$) of massive test particle under the unbalanced external net force. The tensor, $\Omega(\varrho)$, can be written in the form

$$\Omega(\varrho) = D(\varrho)\, \mathcal{Y}(\varrho) \quad (\Omega^{\underline{m}}_{\underline{n}}(\varrho) = D^{\underline{m}}_{\underline{\mu}}(\varrho)\mathcal{Y}^{\underline{\mu}}_{\underline{n}}(\varrho)), \tag{2}$$

provided with the invertible distortion matrix $D\,(D^{\underline{m}}_{\underline{\mu}})$ and the tensor $\mathcal{Y}\,(\mathcal{Y}^{\underline{\mu}}_{\underline{n}} = \partial\, \widetilde{x}^{\underline{\mu}}/\partial\, \underline{x}^{\underline{n}})$. The principle foundation of a world-deformation tensor comprises the following two steps. 1) The basis vectors $\underline{e}_{\,(m)}$, at any point $p \in \underline{M}_2$ is undergone the deformation transformations by means of the matrix $D(\varrho)$:

$$e_{\underline{\mu}}(\varrho) = D^{\underline{m}}_{\underline{\mu}}(\varrho) e_{\underline{m}}. \tag{3}$$

2) A diffeomorphism

$$\widetilde{x}^{\underline{\mu}}(\underline{x}) : \underline{M}_2 \to \mathcal{M}_2 \tag{4}$$

is constructed by seeking a new holonomic coordinates $\widetilde{x}^{\underline{\mu}}(\underline{x})$ as the solutions of the first-order partial differential equations:

$$e_{\underline{\mu}}\, \mathcal{Y}^{\underline{\mu}}_{\underline{m}} = \Omega_{\underline{m}}{}^{\underline{n}}\, \underline{e}_{\underline{n}}, \tag{5}$$

where the conditions of integrability,

$$\partial_{\underline{m}}\, \mathcal{Y}^{\underline{\mu}}_{\underline{n}} = \partial_{\underline{n}}\, \mathcal{Y}^{\underline{\mu}}_{\underline{m}}, \tag{6}$$

and non-degeneracy,

$$det|\mathcal{Y}^{\underline{\mu}}_{\underline{m}}| \neq 0, \tag{7}$$

necessarily hold (Pontryagin, 1984, et al., 1986). Therefore, the $\vartheta \equiv d\underline{x}^{\underline{m}}$ is undergone the following deformation transformations:

$$\vartheta^{\underline{\mu}} = \mathcal{Y}^{\underline{\mu}}_{\underline{m}}\, \underline{\vartheta}^{\underline{m}} = \Omega_{\underline{m}}{}^{\underline{n}} \;< e^{\underline{\mu}}, \underline{e}_{\underline{n}} >\; \underline{\vartheta}^{\underline{m}}. \tag{8}$$

The deformation (1) is composed of the two subsequent deformations

$$\overset{\circ}{\Omega} : \underline{M}_2 \to \underline{V}_2 \tag{9}$$

and

$$\breve{\Omega} : \underline{V}_2 \to \mathcal{M}_2, \tag{10}$$

where $\underline{V}_2$ is the 2D semi-Riemannian space, $\overset{\circ}{\Omega}$ and $\breve{\Omega}$ are the corresponding world deformation tensors. In what follows, we consider the simple spacetime deformation map,

$$\Omega(\varrho) : \underline{M}_2 \to \underline{V}_2 \quad (\Omega = \overset{\circ}{\Omega},\ \breve{\Omega}^{\mu}{}_{\nu} \equiv \delta^{\mu}_{\nu}). \tag{11}$$

The quantities denoted by wiggles here refer to $\underline{V}_2$, but the quantities referring to flat $\underline{M}_2$ space are left, as before, without wiggles. In this case the norm of the infinitesimal displacement on the general smooth differential 2D-manifold $\underline{V}_2$ can be written in terms of the anholonomic spacetime structures:

$$i\tilde{d}(\varrho) = \Omega_b{}^a(\varrho)\, \underline{e}_{\,a}\underline{e}^b \equiv \pi_b{}^{\tilde{c}}(\varrho)\, \pi_{\tilde{c}}{}^a(\varrho)\, \underline{e}_{\,a}\underline{e}^b \;\in\; \underline{V}_2. \tag{12}$$

The matrices, $\pi(\underline{\widetilde{x}})(\varrho) := (\pi_{\tilde{c}}{}^a)(\varrho)$, yield local tetrad deformations

$$e_{\tilde{c}}(\varrho) = \pi_{\tilde{c}}{}^a(\varrho)\, \underline{e}_a, \text{ and } e^{\tilde{c}}(\varrho) = \pi^{\tilde{c}}{}_b(\varrho)\, \underline{e}^b. \tag{13}$$

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They are referred to as the *first deformation matrices*, while the matrices

$$\gamma_{\tilde{c}\tilde{d}}(\underset{\sim}{x}) = {}^*o_{ab}\,\pi_{\tilde{c}}{}^{a}(\underset{\sim}{x})\,\pi_{\tilde{d}}{}^{b}(\underset{\sim}{x}), \tag{14}$$

are *second deformation matrices*. The matrices,

$$\pi^{a}{}_{\tilde{c}}(\underset{\sim}{x}) \in GL(2,R)\,\forall\,\tilde{x}, \tag{15}$$

in general, give rise to right cosets of the Lorentz group, i.e. they are the elements of the quotient group

$$GL(2,R)/SO(1,1), \tag{16}$$

because the Lorentz matrices, L^{r}_{s}, $(r,s=1,0)$ leave the Minkowski metric invariant. A right-multiplication of $\pi_{\tilde{c}}{}^{a}(\underset{\sim}{x})$ by a Lorentz matrix gives an other deformation matrix.

The invertible distortion matrix $D(\varrho)$ is given by a constitutive ansätz:

$$D(\varrho) = \begin{pmatrix} 1 & -\varrho\underline{v}^{(-)} \\ \varrho\underline{v}^{(+)} & 1 \end{pmatrix}, \tag{17}$$

where $\underline{\mu} = (\tilde{\pm})$, $\underline{m} = (\pm)$. These transformations imply a violation $(e^{2}_{(\tilde{\pm})}(\varrho) \neq 0)$ of the condition $(\underline{e}^{\,2}_{(\pm)} = 0)$ of null vectors. The components of metric tensor in $\underline{V}_2$, by virtue of (17), read

$$\begin{array}{l} g_{\tilde{0}\tilde{0}} = (1+\frac{\varrho v^{\underline{1}}}{\sqrt{2}})^2 - \frac{\varrho^2}{2}, \quad g_{\tilde{1}\tilde{1}} = -(1-\frac{\varrho v^{\underline{1}}}{\sqrt{2}})^2 + \frac{\varrho^2}{2}, \\ g_{\tilde{1}\tilde{0}} = g_{\tilde{0}\tilde{1}} = -\sqrt{2}\varrho. \end{array} \tag{18}$$

In general, we parameterize the world-deformation tensor with parameters ν_1 and ν_2 as follows:

$$\begin{array}{l} \Omega_{(+)}^{\;(+)} = \Omega_{(-)}^{\;(-)} = \nu_1(1+\nu_2\,\overline{\varrho}^2), \\ \Omega_{(+)}^{\;(-)} = -\nu_1(1-\nu_2)\varrho\underline{v}^{(-)}, \\ \Omega_{(-)}^{\;(+)} = \nu_1(1-\nu_2)\varrho\underline{v}^{(+)}, \end{array} \tag{19}$$

where $\overline{\varrho}^2 = \underline{v}^2\varrho^2$, $\underline{v}^2 = \underline{v}^{(+)}\underline{v}^{(-)} = 1/2\gamma^2_{\underline{1}}$, and $\gamma_{\underline{1}} = (1-(v^{\underline{1}})^2)^{-1/2}$. The relation (8) can then be recast in an alternative form

$$\vartheta = \nu_1 \begin{pmatrix} 1 & -\nu_2\varrho\,\underline{v}^{(+)} \\ \nu_2\varrho\,\underline{v}^{(-)} & 1 \end{pmatrix} \underline{\vartheta}. \tag{20}$$

The transformation equation for the coordinates, according to (20), becomes

$$\vartheta^{(\tilde{\pm})} = \nu_1\,(\underline{\vartheta}^{(\pm)} \mp \nu_2\,\varrho\underline{v}^{(\pm)}\underline{\vartheta}^{(\mp)}) = \nu_1\,(\underline{v}^{(\pm)} \mp \nu_2\,\varrho\underline{v}^{2})d\underline{x}^{\underline{0}}, \tag{21}$$

which in turn yields the general transformation equations for spatial and temporal coordinates. The latter give a reasonable change at low velocities $\underline{v}^{\underline{1}} \simeq 0$, as

$$d\widetilde{\underline{x}^{\underline{0}}} = \nu_1\,d\underline{x}^{\underline{0}}, \quad d\underline{\widetilde{x}}^{\underline{1}} \simeq \nu_1\,(d\underline{x}^{\underline{1}} - \frac{\nu_2\varrho}{\sqrt{2}}d\underline{x}^{\underline{0}}). \tag{22}$$

In high velocity limit

$$\underline{v}^{\underline{1}} \simeq 1, \quad \overline{\varrho} \simeq 0, \quad d\underline{x}^{(-)} = \underline{v}^{(-)}d\underline{x}^{\underline{0}} \simeq 0,\, \underline{v}^{(+)} \simeq \underline{v} \simeq \sqrt{2}, \tag{23}$$

we have

$$d\widetilde{\underline{x}^{\underline{0}}} = \nu_1\,d\underline{x}^{\underline{0}} \simeq \nu_1\,d\underline{x}^{\underline{1}} \simeq d\underline{\widetilde{x}}^{\underline{1}}. \tag{24}$$

To this end, *the inertial effects become zero.*

Our idea here is this. Suppose a second observer, who makes measurements using a frame of reference $\widetilde{S}_{(2)}$ which is held stationary in $\underline{V}_2$, uses for the test particle the spacetime coordinates $\underline{\widetilde{x}}^{r}(\widetilde{x}^{\underline{0}}, \widetilde{x}^{\underline{1}})$. Then the norm of the infinitesimal displacement on $\underline{V}_2$ can be rewritten as

$$i\widetilde{d} \equiv d\underline{\widetilde{s}} = \widetilde{e}_{\underline{0}}d\widetilde{x}^{\underline{0}} + \widetilde{e}_{\underline{1}}d\widetilde{x}^{\underline{1}}, \tag{25}$$

where $\widetilde{e}_{\underline{0}}$ and $\widetilde{e}_{\underline{1}}$ are, respectively, the temporal and spatial basis vectors. The difference of the line elements $d\underline{s} \in \underline{M}_2$ and $d\underline{\widetilde{s}} \in \underline{V}_2$ can be interpreted in naive way by the second observer that he is subject to gravity,

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so that he thinks he is in the curved space which is due to the deformation of flat space $\underline{M}_2$. However, this difference with equal justice can be reinterpreted by him as a definite criterion for the character of his own state of being in the *absolute* accelerated local non-inertial frame in $\underline{M}_2$(the *absolute acceleration* against $\underline{M}_2$, in Newton's terminology), rather than to any quality of a deformation of $\underline{M}_2$. That is, the (22) becomes conventional transformation equations to accelerated ($a_{net} \neq 0$) axes if we assume $d(\nu_2 \varrho)/\sqrt{2}d\underline{x}^0 = a_{net}$ and $\nu_1(\underline{v}_1 \simeq 0) = 1$, where a_{net} is the magnitude of proper net acceleration. We may calculate a magnitude a_{net} from the embedding relations (97), by considering a test particle accelerated in M_4, under an unbalanced net force other than gravitational.

Recall that the basic distinction between an accelerated observer moving in Minkowski spacetime along an arbitrarily accelerated world line and a momentarily comoving inertial observer is the existence of acceleration scales associated with the noninertial observer. These scales are intrinsic measures of the rate of variation of the local reference frame of the observer along the accelerated path. From a mathematical standpoint, the locality postulate in effect replaces the world line of the observer at each instant by its tangent at that event. Geometrically, the tangent line is the first Frenet approximation to the curve. A frame that undergoes linear and rotational acceleration may be described by the Frenet-Serret frame. This method of moving frames for world lines has been comprehensively discussed in Synge (1960). Each fundamental inertial observer is naturally endowed with an orthonormal tetrad frame that consists of the four unit basis vectors of the Lorentz frame in which the observer is at rest. To be sure, the local orthonormal frame is identical, except perhaps locality, up to a spatial rotation, to the one carried by the instantaneously comoving inertial observer as a consequence of the hypothesis of locality. To describe the acceleration scales mathematically, following Misner et al. (1973), Synge (1960), let us define an orthonormal frame $e_{\hat{a}}$ ($\hat{a} = (\hat{0}, \hat{1}, \hat{2}, \hat{3})$), carried by an accelerated observer, who moves with proper linear 3-acceleration and $\vec{a}(s)$ and proper 3-rotation $\vec{\omega}(s)$, s being the proper time. The time axis must be the time axis of a comoving inertial frame in which the observer is momentarily at rest, i.e. the zeroth leg of the frame $e_{\hat{0}}$ be 4-velocity $\mathbf{u}$ of the observer that is tangent to the worldline at a given point P. The remaining spatial triad frame vectors $e_{\hat{i}}$, orthogonal to $e_{\hat{0}}$, are also parameterized by (s). The spatial triad $e_{\hat{i}}$ rotates with proper 3-rotation $\vec{\omega}(s)$. The 4-velocity vector naturally undergoes Fermi-Walker transport along the curve C, which guarantees that $e_{\hat{0}}(s)$ will always be tangent to C determined by $x^l = x^l(s)$:

$$\frac{de_{\hat{a}}}{ds} = -\Omega\, e_{\hat{a}} \tag{26}$$

where the inertial accelerations are represented by a second rank antisymmetric tensor under global $SO(3,1)$ transformations that is coordinate independent. The antisymmetric rotation tensor Ω splits into a Fermi-Walker transport part Ω_{FW} and a spatial rotation part Ω_{SR} (the latter is in fact the rotational frequency of the frame):

$$\Omega^{lk}_{FW} = a^l u^k - a^k u^l, \quad \Omega^{lk}_{SR} = u_m \omega_n \varepsilon^{mnlk}. \tag{27}$$

The relative rotational acceleration of a Frenet- Serret frame with respect to a Fermi-Walker transported frame is taken to characterize important phenomena, like the gyroscopic precession. Of course the proper interpretation of the translational and rotational accelerations of a frame makes sense at least in the case of asymptotically flat spacetimes. The 4-vector of rotation ω^l is orthogonal to 4-velocity u^l, therefore, in the rest frame it becomes $\omega^l(0, \vec{\omega})$, and ε^{mnlk} is the Levi-Civita tensor with $\varepsilon^{0123} = -1$. The spacelike hyperplane $S(P)$ is associated to each point P on the world line of the observer orthogonal to it. Defining $x^0 = t = s$ and x^1, x^2, x^3 as Cartesian coordinates using the triad $e_{\hat{i}}(s)$ with the observer at the origin, $x^\mu = (x^0, x^1, x^2, x^3)$ are the local coordinates for the observer. The tetrad $e_{\hat{\mu}}(s)$ can be parallel transported from P to all neighboring points on $S(P)$. This defines the orthonorrnal tetrad field $e_\mu(x^\nu)$. Thus the $\vec{a}_{net}$ will be a local net 3-acceleration of an arbitrary observer with proper linear 3 - acceleration $\vec{a}$ and proper 3-angular velocity $\vec{\omega}$ in M_4 measured in the rest frame $(0, \vec{a}_{net})$

$$\vec{a}_{net} = \frac{d\vec{u}}{ds} = \vec{a} \wedge \vec{u} + \vec{\omega} \times \vec{u}. \tag{28}$$

A magnitude of $\vec{a}_{net}$ can be computed as the simple invariant of the absolute value $|\frac{d\mathbf{u}}{ds}|$ as measured in the rest frame: $a_{net} = |\mathbf{a}| = |\frac{d\mathbf{u}}{ds}|$. Then the very concept of the local *absolute acceleration* can be introduced using the Fermi-Walker transported frames

$$\vec{a}_{abs} := \vec{e}_{\underline{1}} \frac{d(\nu_2 \varrho)}{\sqrt{2}d\underline{s}} = \vec{e}_{\underline{1}}\, \underline{a} = \vec{n}\, |\mathbf{a}|, \tag{29}$$

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where the axis $\vec{e}_{\underline{1}}$ of the system $S_{(2)}$, according to embedding map (94), lies along the net 3-acceleration. In particular case of $\nu_1 = \nu_2 = 1$, the world-deformation tensor is simplified

$$\Omega_{\underline{m}}^{\underline{n}} = \Omega(\overline{\varrho})\delta_{\underline{m}}^{\underline{n}}, \quad \Omega(\overline{\varrho}) = 1 + \overline{\varrho}^2, \tag{30}$$

so the deformed line element becomes $d\underline{\widetilde{s}}^2 = \Omega^2(\overline{\varrho})\, d\underline{s}^2$. The (21) gives

$$\frac{d^2\underline{\widetilde{x}}^{(\pm)}}{d\underline{\widetilde{s}}^2} = \mp \frac{1}{2\Omega^2(\overline{\varrho})\gamma_{\underline{1}}^2} \frac{d(\varrho)}{d\underline{s}}. \tag{31}$$

We set the value of the acceleration, $\underline{a}_{in}^{\underline{1}} = \frac{d^2\underline{\widetilde{x}}^1}{d\underline{\widetilde{s}}^2}$ of observer in $\underline{V}_2$, equal to

$$\underline{a}_{in}^{\underline{1}} = -\Gamma^{\underline{1}}_{\underline{\tilde{\mu}}\underline{\tilde{\nu}}}(\overline{\varrho}) \frac{d\underline{\widetilde{x}}^{\tilde{\mu}}}{d\underline{\widetilde{s}}} \frac{d\underline{\widetilde{x}}^{\tilde{\nu}}}{d\underline{\widetilde{s}}}, \tag{32}$$

where $\Gamma^{\underline{1}}_{\underline{\tilde{\mu}}\underline{\tilde{\nu}}}(\overline{\varrho})$ are the Christoffel symbols constructed by the metric (18). As a consequence of the so-called *inertial acceleration* (32), an observer in $\underline{V}_2$ obeys *exact* WPE, i.e. it defines a locally inertial reference frame in free fall that moves freely under the action of *fictitious* gravitational forces along the geodesic. Then

$$\underline{a}_{in}^{\underline{1}} = \frac{1}{\sqrt{2}} \left(\frac{d^2\underline{\widetilde{x}}^{(+)}}{d\underline{\widetilde{s}}^2} - \frac{d^2\underline{\widetilde{x}}^{(-)}}{d\underline{\widetilde{s}}^2} \right) = -\frac{1}{\Omega^2(\overline{\varrho})\gamma_{\underline{1}}} \frac{d(\varrho)}{\sqrt{2}d\underline{s}}, \tag{33}$$

This, combined with (29) and embedding relations for accelerations (97),

$$\underline{a}_{in}^{\underline{1}} = a_{in} = |\vec{a}_{in}|, \tag{34}$$

yield a relationship of the magnitudes of *absolute* and *inertial* accelerations in M_4,

$$\Omega^2(\overline{\varrho})\, \gamma_{\underline{1}}\, a_{in} = -\frac{d\varrho}{\sqrt{2}d\underline{s}} = -a_{abs}. \tag{35}$$

Thus we seen that a general deformation of MS_p is the origin of the local *absolute* ($\vec{a}_{abs}$) and *inertial* ($\vec{a}_{in}$) accelerations that satisfy the key relation

$$\Omega^2(\overline{\varrho})\, \gamma_{\underline{1}}\, \vec{a}_{in} = -\vec{a}_{abs}. \tag{36}$$

In non-relativistic case of low velocities, if $\vec{a}_{abs}$ is due solely to Newtonian gravity in the weak field limit, the (36) furnishes a justification for the introduction of the WPE in M_4:

$$\vec{a}_{in} = -\vec{a}_{abs}, \tag{37}$$

namely, the local effects of gravity are not measured by an observer in free fall, who defines a locally inertial reference frame. In the following sections we will look at inertial effects arising from (36). For a justification of the relativistic WPE in the semi-Riemannian space of an arbitrary gravitational field, see Section 5.

Now by means of (114) and (115), we have to write the rate, ϱ, in terms of the Lorentz spinors $(\underline{\theta}, \underline{\bar{\theta}})$, and period of superoscillations (τ):

$$\varrho(\underline{\theta}, \underline{\bar{\theta}}, \tau) = \sqrt{2}(\underline{\widetilde{v}} - \sqrt{2}v_c) = 2\sqrt{2}(\underline{\theta}_1\, \underline{\bar{\theta}}_1 \underline{\theta}_2\, \underline{\bar{\theta}}_2)^{1/2} \frac{d\tau}{d\underline{\widetilde{s}}}. \tag{38}$$

A period of superoscillations, $\tau(\underline{\widetilde{s}})$, can then be determined as a function of proper time $(\underline{\widetilde{s}})$ induced by the *world-deformation* tensor $\Omega(\underline{\widetilde{s}})$:

$$\tau(\underline{\widetilde{s}}) = 4^{-1}(\underline{\theta}_1\, \underline{\bar{\theta}}_1 \underline{\theta}_2\, \underline{\bar{\theta}}_2)^{-1} \int_0^{\underline{\widetilde{s}}} \sqrt{\Omega(\underline{\widetilde{s}}') - 1}\, d\underline{\widetilde{s}}'. \tag{39}$$

So that the magnitudes of the net and absolute accelerations induced by the $\Omega(\underline{\widetilde{s}})$ read

$$a_{net} = \sqrt{2}v_c\, a_{abs} = \frac{d}{d\underline{s}} \sqrt{\Omega(\underline{\widetilde{s}}) - 1}. \tag{40}$$

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3. Model building in background M_4

The (36) provides a quantitative means for the *inertial force* $\vec{f}_{(in)}$:

$$\vec{f}_{(in)} = m\vec{a}_{in} = -\frac{m\,\vec{a}_{abs}}{\Omega^2(\overline{\varrho})\,\gamma_{\underline{1}}} = -m\,\vec{a}_{abs}\frac{\sqrt{2}\underline{v}_c}{(1+\overline{\varrho}^2)^2}, \tag{41}$$

where $\overline{\varrho} = 2\underline{v}_c^2(d\tau/d\widetilde{\underline{s}}) = (1/\gamma_{\underline{1}}^2)(d\tau/d\widetilde{\underline{s}})$. In case of absence of rotation, we may write the local *absolute acceleration* (29) in terms of the relativistic force f^l acting on a particle with coordinates $x^l(s)$:

$$f^l(f^0,\,\vec{f}) = m\frac{d^2x^l}{ds^2} = L^l_k(\vec{v})F^k. \tag{42}$$

Here $F^k(0,\,\vec{F})$ is the force defined in the rest frame of the test particle, $L^l_k(\vec{v})$ is the Lorentz transformation matrix $(i,j = 1,2,3)$:

$$L^i_j = \delta_{ij} - (\gamma - 1)\frac{v_i v_j}{|\vec{v}|^2}, \quad L^0_i = \gamma v_i, \tag{43}$$

where $\gamma = (1-\vec{v}^2)^{-1/2} = \gamma_{\underline{1}}$. So

$$|\mathbf{a}| = \frac{1}{m}|f^l| = \frac{1}{m}(f^l f_l)^{1/2} = \frac{1}{m\gamma}|\vec{f}|, \tag{44}$$

and hence the (41) and (44) give

$$\vec{f}_{(in)} = -\frac{\sqrt{2}\underline{v}_c}{\gamma(1+\overline{\varrho}^2)^2}[\vec{F} + (\gamma-1)\vec{n}(\vec{n}\cdot\vec{F})]. \tag{45}$$

At low velocities $|\vec{v}| \simeq 0$ and tiny accelerations $\frac{d\tau}{d\underline{s}} \to 0$ we usually experience, one has $\underline{v}_c(\underline{\theta},\bar{\underline{\theta}},\tau \simeq const) \simeq 0$, and that $\Omega(\overline{\varrho}) \simeq 1$, the (45) is reduced to the conventional non-relativistic law of inertia for a classical particle motion

$$\vec{f}_{(in)} = -m\vec{a}_{abs} = -\vec{F}. \tag{46}$$

At high velocities $|\vec{v}| \simeq 1$, if $(\vec{n}\cdot\vec{F}) \neq 0$, the inertial force (45) becomes

$$\vec{f}_{(in)} \simeq -\frac{1}{(1+\overline{\varrho}^2)^2\,\gamma}\vec{n}(\vec{n}\cdot\vec{F}), \tag{47}$$

and it vanishes in the limit of the photon ($|\vec{v}| = 1$, $\overline{\varrho}^2 = \gamma^{-4}(d\tau/d\widetilde{\underline{s}}) \to 0$, $m = 0$). Thus inertial reference systems are specified for classical particle motions as relative, while we do not need to specify a reference system for the propagation of light, i.e. its movement is completely independent of the motion of inertial observers and it can thus be considered as absolute.

Hence, we see that a deformation of $\underline{M}_2$ is the cause of arising the *absolute* and *inertial* accelerations, and the *inertial* force. Whereas the *relative* acceleration of a particle in 4D Minkowski space, M_4 (both magnitude and direction), to the contrary, has nothing to do with a deformation of $\underline{M}_2$ and, thus, it cannot produce the inertia effects.

4. Beyond the hypothesis of locality

In standard framework of SR, an assumption is required for the construction of reference frame of an accelerated observer to relate the ideal inertial observers to actual observers that are all noninertial, i.e., accelerated. Therefore, it is a long-established practice in physics to use the hypothesis of locality, see e.g. (Maluf & Faria, 2008, Maluf et al., 2007, Marzlin, 1996, Mashhoon, 2002, 2011, Misner et al., 1973) and references therein, for extension of the Lorentz invariance to accelerated observers in Minkowski space-time. The geometrical structures, referred to a noninertial coordinate frame of accelerating and rotating observer in Minkowski space-time, were computed on the base of the assumption that an accelerated observer is pointwise inertial, which in effect replaces an accelerated observer at each instant with a momentarily comoving inertial observer along its wordline. This assumption is known to be an approximation limited to motions with sufficiently low accelerations, which works out because all relevant length scales in feasible experiments are very small in relation to the huge acceleration lengths of the tiny accelerations we usually experience, therefore, the curvature of the wordline could be ignored and that the differences between observations by accelerated and comoving inertial observers will also be very small. However, it seems quite clear that such an approach is a work in progress, which reminds us of a puzzling underlying reality of

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inertia, and that it will have to be extended to describe physics for arbitrary accelerated observers. Ever since this question has become a major preoccupation of physicists. The hypothesis of locality represents strict restrictions, because it approximately replaces a noninertial frame of reference $\widetilde{S}_{(2)}$, which is held stationary in the deformed space $\mathcal{M}_2 \equiv \underline{V}_2^{(\varrho)}\,(\varrho \neq 0)$, where $\underline{V}_2$ is the 2D semi-Riemannian space, with a continuous infinity set of the inertial frames $\{S_{(2)}, S'_{(2)}, S''_{(2)}, ...\}$ given in the flat $\underline{M}_2\,(\varrho = 0)$. In this situation the use of the hypothesis of locality is physically unjustifiable. Therefore, it is worthwhile to go beyond the hypothesis of locality with special emphasis on deformation $\underline{M}_2 \longrightarrow \underline{V}_2^{(\varrho)}$, which we might expect will essentially improve the standard framework.

The notation will be slightly different from the previous section. Namely, the orthonormal frame $e_{\hat{a}}$ (26), carried by an accelerated observer, will be denoted by wiggles such that

$$\widetilde{e}_{\hat{a}} = \overline{e}^{\,\mu}_{\;\hat{a}}\,\overline{e}_{\mu} = \widetilde{e}^{\,\mu}_{\;\hat{a}}\,\widetilde{e}_{\mu}, \quad \widetilde{\vartheta}^{\hat{b}} = \overline{e}_{\mu}^{\;\hat{b}}\,\overline{\vartheta}^{\mu} = \widetilde{e}_{\mu}^{\;\hat{b}}\,\widetilde{\vartheta}^{\mu}, \tag{48}$$

with $\overline{e}_{\mu} = \partial_{\mu} = \partial/\partial x^{\mu}$, $\quad \widetilde{e}_{\mu} = \widetilde{\partial}_{\mu} = \partial/\partial\widetilde{x}^{\mu}$, $\quad \overline{\vartheta}^{\mu} = dx^{\mu}$, $\quad \widetilde{\vartheta}^{\mu} = d\widetilde{x}$.

Following (Mashhoon, 2002, Misner et al., 1973), let us to introduce a geodesic coordinate system - the coordinates relative to the accelerated observer (the laboratory coordinates), which is in general valid in a sufficiently narrow worldtube along the timelike worldline of the observer. The coframe members $\{\widetilde{\vartheta}^{\hat{b}}\}$ are the objects of dual counterpart: $\widetilde{e}_{\hat{a}} \rfloor \widetilde{\vartheta}^{\hat{b}} = \delta^{b}_{a}$. We choose the zeroth leg of the frame, $\widetilde{e}_{\hat{0}}$, as before, to be the unit vector $\mathbf{u}$ that is tangent to the worldline at a given event $x^{\mu}(s)$, where (s) is a proper time measured along the accelerated path by the standard (static inertial) observers in the underlying global inertial frame. The condition of orthonormality for the frame field $\overline{e}^{\,\mu}_{\;\hat{a}}$ reads

$$\eta_{\mu\nu}\,\overline{e}^{\,\mu}_{\;\hat{a}}\,\overline{e}^{\,\nu}_{\;\hat{b}} = o_{\hat{a}\hat{b}} = diag(+ - - -). \tag{49}$$

A tetrad field $\overline{e}^{\,\mu}_{\;\hat{a}}$ allows the projection of vectors and tensors in spacetime in the local frame of an observer (Maluf & Faria, 2008, Maluf et al., 2007). Each set of tetrad fields defines a class of reference frames (Hehl et al., 1991). In (Maluf & Faria, 2008) a simple recipe is given for obtaining a Fermi-Walker transported frame out of any set of tetrad fields, for an arbitrary spacetime, assuming that the frame is transported along a timelike trajectory. Reference frames may be characterized by an antisymmetric acceleration tensor Φ_{ab}, whose components are identified as the inertial accelerations of the frame (the translational acceleration and the frequency of rotation of the frame).

In analogy with the Faraday tensor (Maluf & Faria, 2008, Maluf et al., 2007, Marzlin, 1996, Mashhoon, 2002, 2011), one can identify the antisymmetric acceleration tensor

$$\Phi_{ab} \longrightarrow (-\mathbf{a},\, \omega), \tag{50}$$

with $\mathbf{a}(s)$ as the translational acceleration

$$\Phi_{0i} = -a_i, \tag{51}$$

and $\omega(s)$ as the frequency of rotation of the local spatial frame with respect to a nonrotating (Fermi- Walker transported) frame

$$\Phi_{ij} = -\varepsilon_{ijk}\,\omega^{k}. \tag{52}$$

The invariants constructed out of Φ_{ab} establish the acceleration scales and lengths. The hypothesis of locality holds for huge proper acceleration lengths $|I|^{-1/2} \gg 1$ and $|I^{*}|^{-1/2} \gg 1$, where the scalar invariants are given by $I = (1/2)\,\Phi_{ab}\,\Phi^{ab} = -\vec{a}^{2} + \vec{\omega}^{2}$ and $I^{*} = (1/4)\,\Phi^{*}_{ab}\,\Phi^{ab} = -\vec{a}\cdot\vec{\omega}$ ($\Phi^{*}_{ab} = \varepsilon_{abcd}\,\Phi^{cd}$) (Mashhoon, 2002, 2011). Suppose the displacement vector $z^{\mu}(s)$ represents the position of the accelerated observer. According to the hypothesis of locality, at any time (s) along the accelerated worldline the hypersurface orthogonal to the worldline is Euclidean space and we usually describe some event on this hypersurface ("local coordinate system") at x^{μ} to be at $\widetilde{x}^{\mu}$, where x^{μ} and $\widetilde{x}^{\mu}$ are connected via $\widetilde{x}^{\,0} = s$ and

$$x^{\mu} = z^{\mu}(s) + \widetilde{x}^{\,i}\,\overline{e}^{\,\mu}_{\;\hat{i}}(s). \tag{53}$$

The standard metric of semi-Riemannian 4D background space $V_4^{(0)}$ in noninertial system of the accelerating and rotating observer, computed on this base. The hypothesis of locality leads to the 2D semi-Riemannian space, $V_{\underline{2}}^{(0)}$, with the incomplete metric $\widetilde{g}(\varrho = 0)$:

$$\widetilde{g} = \left[(1 + \widetilde{x}^{\,\underline{1}}\widetilde{\varphi}_0)^2 - (\widetilde{x}^{\,\underline{1}}\widetilde{\varphi}_1)^2\right]\, d\widetilde{x}^{\,\underline{0}} \otimes d\widetilde{x}^{\,\underline{0}} - 2\,(\widetilde{x}^{\,\underline{1}}\widetilde{\varphi}_1)\, d\widetilde{x}^{\,\underline{1}} \otimes d\widetilde{x}^{\,\underline{0}} - d\widetilde{x}^{\,\underline{1}} \otimes d\widetilde{x}^{\,\underline{1}}, \tag{54}$$

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provided,

$$\widetilde{x}^{\underline{1}}\widetilde{\varphi}_0 = \widetilde{x}^{\,i}\,\Phi^0_i, \quad \widetilde{x}^{\underline{1}}\widetilde{\varphi}_1 = \widetilde{x}^{\,i}\,\Phi^j_i\,\widetilde{e}^{\,-1}_j. \tag{55}$$

Therefore, our strategy now is to deform the metric (54) by carrying out an additional deformation of semi-Riemannian 4D background space

$$V^{(0)}_4 \longrightarrow V^{(\varrho)}_4, \tag{56}$$

in order it becomes on the same footing with the complete metric $\widetilde{g}$ $\;(\varrho \neq 0)$ (18) of the distorted space $\underline{V}^{(\varrho)}_2$. Let the Latin letters $\hat{r}, \hat{s}, ... = 0, 1$ be the anholonomic indices referred to the anholonomic frame $e_{\hat{r}} = e^{\tilde{s}}{}_{\hat{r}}\,\partial_{\tilde{s}}$, defined on the $\underline{V}^{(\varrho)}_2$, with $\partial_{\tilde{s}} = \partial/\partial\,\widetilde{x}^{\underline{s}}$ as the vectors tangent to the coordinate lines. So, a smooth differential 2D-manifold $\underline{V}^{(\varrho)}_2$ has at each point $\widetilde{x}^{\underline{s}}$ a tangent space $\widetilde{T}_{\underline{\widetilde{x}}}\underline{V}^{(\varrho)}_2$, spanned by the frame, $\{e_{\hat{r}}\}$, and the coframe members $\vartheta^{\hat{r}} = e_{\underline{s}}{}^{\hat{r}}\,d\widetilde{x}^{\underline{s}}$, which constitute a basis of the covector space $\widetilde{T}^{\star}_{\underline{\widetilde{x}}}\underline{V}^{(\varrho)}_2$. All this nomenclature can be given for $\underline{V}^{(0)}_2$ too. Then, we may compute corresponding vierbein fields $\widetilde{e}_r{}^{\hat{s}}$ and $e_r{}^{\hat{s}}$ from the equations

$$g_{\tilde{r}\tilde{s}} = \widetilde{e}_{\tilde{r}}{}^{\hat{r}'}\,\widetilde{e}_{\tilde{s}}{}^{\hat{s}'}\,o_{\hat{r}'\hat{s}'}, \quad g_{\tilde{r}\tilde{s}}(\varrho) = e_{\tilde{r}}{}^{\hat{r}'}(\varrho)\,e_{\tilde{s}}{}^{\hat{s}'}(\varrho)\,o_{\hat{r}'\hat{s}'}, \tag{57}$$

with $\widetilde{g}_{rs}$ (54) and $g_{\tilde{r}\tilde{s}}(\varrho)$ (18). Hence

$$\begin{array}{l} \widetilde{e}_{\tilde{0}}{}^{\hat{0}} = 1 + \vec{a}\cdot\vec{\widetilde{x}}, \quad \widetilde{e}_{\tilde{0}}{}^{\hat{1}} = \vec{\omega}\wedge\vec{\widetilde{x}}, \quad \widetilde{e}_{\tilde{1}}{}^{\hat{0}} = 0, \quad \widetilde{e}_{\tilde{1}}{}^{\hat{1}} = 1, \\ e_{\tilde{0}}{}^{\hat{0}}(\varrho) = 1 + \frac{\varrho v_{\underline{1}}}{\sqrt{2}}, \quad e_{\tilde{0}}{}^{\hat{1}}(\varrho) = \frac{\varrho}{\sqrt{2}}, \quad e_{\tilde{1}}{}^{\hat{0}}(\varrho) = -\frac{\varrho}{\sqrt{2}}, \quad e_{\tilde{1}}{}^{\hat{1}}(\varrho) = 1 - \frac{\varrho v_{\underline{1}}}{\sqrt{2}}. \end{array} \tag{58}$$

A deformation (56) is equivalent to a straightforward generalization of (53) as

$$x^{\mu} \;\longrightarrow\; x^{\mu}_{(\varrho)} = z^{\mu}_{(\varrho)}(s) + \widetilde{x}^{\,i}\,e^{\mu}{}_{\hat{i}}(s), \tag{59}$$

provided, as before, $\widetilde{x}^{\mu}$ denotes the coordinates relative to the accelerated observer in 4D background space $V^{(\varrho)}_4$. A displacement vector from the origin is then

$$dz^{\mu}_{\varrho}(s) = e^{\mu}{}_{\hat{0}}(\varrho)\,d\widetilde{x}^0. \tag{60}$$

Inverting $e_r{}^{\hat{s}}(\varrho)$ (58), we obtain

$$e^{\mu}{}_{\hat{a}}(\varrho) = \pi_{\hat{a}}{}^{\hat{b}}(\varrho)\,\overline{e}^{\,\mu}{}_{\hat{b}}, \tag{61}$$

where

$$\begin{array}{l} \pi^{\hat{0}}_{\hat{0}}(\varrho) \equiv (1 + \frac{\varrho^2}{2\gamma^2_{\underline{1}}})^{-1}(1 - \frac{\varrho v_{\underline{1}}}{\sqrt{2}})\,(1 + \vec{a}\cdot\vec{\widetilde{x}}), \quad \pi^{\hat{i}}_{\hat{0}}(\varrho) \equiv -(1 + \frac{\varrho^2}{2\gamma^2_{\underline{1}}})^{-1}\frac{\varrho}{\sqrt{2}}\,\widetilde{e}^{i}\,(1 + \vec{a}\cdot\vec{\widetilde{x}}), \\ \pi^{\hat{0}}_{\hat{i}}(\varrho) \equiv (1 + \frac{\varrho^2}{2\gamma^2_{\underline{1}}})^{-1}\left[(\vec{\omega}\wedge\vec{\widetilde{x}})(1 - \frac{\varrho v_{\underline{1}}}{\sqrt{2}}) - \frac{\varrho}{\sqrt{2}}\right]\widetilde{e}^{\,-1}_i, \quad \pi^{\hat{j}}_{\hat{i}}(\varrho) = \delta^j_i\,\pi(\varrho), \\ \pi(\varrho) \equiv (1 + \frac{\varrho^2}{2\gamma^2_{\underline{1}}})^{-1}\;\left[(\vec{\omega}\wedge\vec{\widetilde{x}})\,\frac{\varrho}{\sqrt{2}} + 1 + \frac{\varrho v_{\underline{1}}}{\sqrt{2}}\right]. \end{array} \tag{62}$$

Thus,

$$dx^{\mu}_{\varrho} = dz^{\mu}_{\varrho}(s) + d\widetilde{x}^{\,i}\,e^{\mu}{}_{\hat{i}} + \widetilde{x}^{\,i}\,de^{\mu}{}_{\hat{i}}(s) = (\tau^{\hat{b}}\,d\widetilde{x}^0 + \pi^{\hat{b}}_{\hat{i}}\,d\widetilde{x}^{\,i})\,\overline{e}^{\,\mu}{}_{\hat{b}}, \tag{63}$$

where

$$\tau^{\hat{b}} \equiv \pi^{\hat{b}}_{\hat{0}} + \widetilde{x}^{\,i}\left(\pi^{\hat{a}}_{\hat{i}}\Phi^b_a + \frac{d\pi^{\hat{b}}_{\hat{i}}}{ds}\right). \tag{64}$$

Hence, in general, the metric in noninertial frame of arbitrary accelerating and rotating observer in Minkowski space-time is

$$\widetilde{g}(\varrho) = \eta_{\mu\nu}\,dx^{\mu}_{\varrho}\otimes dx^{\nu}_{\varrho} = W_{\mu\nu}(\varrho)\,d\widetilde{x}^{\mu}\otimes d\widetilde{x}^{\nu}, \tag{65}$$

which can be conveniently decomposed according to

$$\begin{array}{l} W_{00}(\varrho) = \pi^2\left[(1 + \vec{a}\cdot\vec{\widetilde{x}})^2 + (\vec{\omega}\cdot\vec{\widetilde{x}})^2 - (\vec{\omega}\cdot\vec{\omega})(\vec{\widetilde{x}}\cdot\vec{\widetilde{x}})\right] + \gamma_{00}(\varrho), \\ W_{0i}(\varrho) = -\pi^2\,(\vec{\omega}\wedge\vec{\widetilde{x}})^i + \gamma_{0i}(\varrho), \quad W_{ij}(\varrho) = -\pi^2\,\delta_{ij} + \gamma_{ij}(\varrho), \end{array} \tag{66}$$

provided,

$$\begin{array}{l} \gamma_{00}(\varrho) = \pi\,\left[(1 + \vec{a}\cdot\vec{\widetilde{x}})\zeta^0 - (\vec{\omega}\wedge\vec{\widetilde{x}})\cdot\vec{\zeta}\right] + (\zeta^0)^2 - (\vec{\zeta})^2, \quad \gamma_{0i}(\varrho) = -\pi\,\zeta^i + \tau^{\hat{0}}\,\pi^{\hat{0}}_{\hat{i}}, \\ \gamma_{ij}(\varrho) = \pi^{\hat{0}}_{\hat{i}}\,\pi^{\hat{0}}_{\hat{j}}, \quad \zeta^0 = \pi\,\left(\tau^{\hat{0}} - 1 - \vec{a}\cdot\vec{\widetilde{x}}\right), \quad \vec{\zeta} = \pi\,\left(\vec{\tau} - \vec{\omega}\wedge\vec{\widetilde{x}}\right). \end{array} \tag{67}$$

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As we expected, according to (65)- (67), the matric $\widetilde{g}(\varrho)$ is decomposed in the following form:

$$g(\varrho) = \pi^2(\varrho)\, \widetilde{g} + \gamma(\varrho), \tag{68}$$

where

$$\gamma(\varrho) = \gamma_{\mu\nu}(\varrho)\, d\widetilde{x}^{\mu} \otimes d\widetilde{x}^{\nu}, \quad \text{and} \quad \Upsilon(\varrho) = \pi^{\hat{a}}_{\hat{a}}(\varrho) = \pi(\varrho). \tag{69}$$

In general, the geodesic coordinates are admissible as long as

$$\left(1 + \vec{a}\cdot\vec{\widetilde{x}} + \tfrac{\zeta^0}{\pi}\right)^2 > \left(\vec{\omega}\wedge\vec{\widetilde{x}} + \tfrac{\vec{\zeta}}{\pi}\right)^2. \tag{70}$$

The equations (54) and (65) say that the vierbein fields, with entries

$$\eta_{\mu\nu}\, \bar{e}^{\mu}{}_{\hat{a}}\, \bar{e}^{\nu}{}_{\hat{b}} = o_{\hat{a}\hat{b}} \quad \text{and} \quad \eta_{\mu\nu}\, e^{\mu}{}_{\hat{a}}\, e^{\nu}{}_{\hat{b}} = \gamma_{\hat{a}\hat{b}}, \tag{71}$$

lead to the relations

$$\widetilde{g} = o_{\hat{a}\hat{b}}\, \widetilde{\vartheta}^{\hat{a}} \otimes \widetilde{\vartheta}^{\hat{b}}, \quad \text{and} \quad g = o_{\hat{a}\hat{b}}\, \vartheta^{\hat{a}} \otimes \vartheta^{\hat{b}} = \gamma_{\hat{a}\hat{b}}\, \widetilde{\vartheta}^{\hat{a}} \otimes \widetilde{\vartheta}^{\hat{b}}, \tag{72}$$

which readily leads to the coframe fields:

$$\begin{aligned} \widetilde{\vartheta}^{\hat{b}} &= \bar{e}_{\mu}{}^{\hat{b}}\, dx^{\mu} = \widetilde{e}^{\hat{b}}{}_{\mu}\, d\widetilde{x}^{\mu}, \quad \widetilde{e}^{\hat{b}}{}_{0} = N^{b}_{0}, \quad \widetilde{e}^{\hat{b}}{}_{i} = N^{b}_{i}, \\ \vartheta^{\hat{b}} &= \bar{e}_{\mu}{}^{\hat{b}}\, dx^{\mu}_{\varrho} = e^{\hat{b}}{}_{\mu}\, d\widetilde{x}^{\mu} = \pi^{\hat{b}}{}_{\hat{a}}\, \widetilde{\vartheta}^{\hat{a}}, \quad e^{\hat{b}}{}_{0} = \tau^{\hat{b}}, \quad e^{\hat{b}}{}_{i} = \pi^{\hat{b}}{}_{\hat{i}}. \end{aligned} \tag{73}$$

Here

$$N^{0}_{0} = N \equiv \left(1 + \vec{a}\cdot\vec{\widetilde{x}}\right), \quad N^{0}_{i} = 0, \quad N^{i}_{0} = N^{i} \equiv \left(\vec{\omega}\cdot\vec{\widetilde{x}}\right)^{i}, \quad N^{j}_{i} = \delta^{j}_{i}. \tag{74}$$

In the standard $(3+1)$-decomposition of space-time, N and N^i are known as *lapse function* and *shift vector*, respectively (Gronwald & Hehl, 1996). Hence, we may easily recover the frame field

$$e_{\hat{a}} = e^{\mu}{}_{\hat{a}}\, \widetilde{e}_{\mu} = \pi_{\hat{a}}{}^{\hat{b}}\, \widetilde{e}_{\hat{b}}, \tag{75}$$

by inverting (73):

$$\begin{aligned} e_{\hat{0}}(\varrho) &= \frac{\pi(\varrho)}{\pi(\varrho)\,\tau^{\hat{0}}(\varrho) - \pi^{\hat{0}}_{\hat{k}}(\varrho)\,\tau^{\hat{k}}(\varrho)}\, \widetilde{e}_0 - \frac{\tau^{\hat{i}}(\varrho)}{\pi(\varrho)\,\tau^{\hat{0}}(\varrho) - \pi^{\hat{0}}_{\hat{k}}(\varrho)\,\tau^{\hat{k}}(\varrho)}\, \widetilde{e}_i, \\ e_{\hat{i}}(\varrho) &= -\frac{\pi^{\hat{0}}_{\hat{i}}(\varrho)}{\pi(\varrho)\,\tau^{\hat{0}}(\varrho) - \pi^{\hat{0}}_{\hat{k}}(\varrho)\,\tau^{\hat{k}}(\varrho)}\, \widetilde{e}_0 + \pi^{-1}(\varrho)\left[\delta^{j}_{i} + \frac{\tau^{j}(\varrho)\,\pi^{\hat{0}}_{\hat{i}}(\varrho)}{\pi(\varrho)\,\tau^{\hat{0}}(\varrho) - \pi^{\hat{0}}_{\hat{k}}(\varrho)\,\tau^{\hat{k}}(\varrho)}\right] \widetilde{e}_j. \end{aligned} \tag{76}$$

A *generalized transport* for deformed frame $e_{\hat{a}}$, which includes both the Fermi-Walker transport and deformation of $\underline{M}_2$, can be written in the form

$$\frac{de^{\mu}{}_{\hat{a}}}{ds} = \widetilde{\Phi}_{a}{}^{b}\, e^{\mu}{}_{\hat{b}}, \tag{77}$$

where a *deformed acceleration tensor* $\widetilde{\Phi}_{a}{}^{b}$ concisely is given by

$$\widetilde{\Phi} = \frac{d\ln\pi}{ds} + \pi\,\Phi\,\pi^{-1}. \tag{78}$$

Thus, we derive the tetrad fields $e_{r}{}^{\hat{s}}(\varrho)$ (58) and $e^{\mu}{}_{\hat{a}}(\varrho)$ (76) as a function of *local rate* ϱ of instantaneously change of a constant velocity (both magnitude and direction) of a massive particle in M_4 under the unbalanced net force, describing corresponding *fictitious graviton*. Therewith the *fictitious gravitino*, $\psi^{\alpha}_{\hat{m}}(\varrho)$, will be arisen under infinitesimal transformations of local supersymmetry.

5. Involving the background semi-Riemann space V_4; Justification for the introduction of the WPE

We can always choose *natural coordinates* $X^{\alpha}(T,X,Y,Z) = (T,\,\vec{X})$ with respect to the axes of the local free-fall coordinate frame $S^{(l)}_4$ in an immediate neighbourhood of any space-time point $(\widetilde{x}_p) \in V_4$ in question of the background semi-Riemannian space, V_4, over a differential region taken small enough so that we can

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neglect the spatial and temporal variations of gravity for the range involved. The values of the metric tensor $\widetilde{g}_{\mu\nu}$ and the affine connection $\widetilde{\Gamma}^{\lambda}_{\mu\nu}$ at the point $(\widetilde{x}_p)$ are necessarily sufficient information for determination of the natural coordinates $X^{\alpha}(\widetilde{x}^{\mu})$ in the small region of the neighbourhood of the selected point Weinberg (1972). Then the whole scheme outlined above will be held in the frame $S_4^{(l)}$. The relativistic gravitational force $f_g^{\mu}(\widetilde{x})$ exerted on the test particle of the mass (m) moving in $\underline{V}_2$ freely under the influence of purely arbitrary gravitational forces along the geodesic is given by

$$f_g^{\mu}(\widetilde{x}) = m\frac{d^2\widetilde{x}^{\mu}}{d\widetilde{s}^2} = -m\Gamma^{\mu}_{\nu\lambda}\frac{d\widetilde{x}^{\nu}}{d\widetilde{s}}\frac{d\widetilde{x}^{\lambda}}{d\widetilde{s}}. \tag{79}$$

The frame $S_4^{(l)}$ will be valid if only the gravitational force given in this coordinate frame

$$f^{\alpha}_{g(l)} = \frac{\partial X^{\alpha}}{\partial\widetilde{x}^{\mu}}f_g^{\mu}, \tag{80}$$

could be removed by the inertial force. Whereas, as before, the two systems S_2 and $S_4^{(l)}$ can be chosen in such a way as the axis $\vec{e}_{\underline{1}}$ of $S_{(2)}$ lies $(\vec{e}_{\underline{1}} = \vec{e}_f)$ along the acting net force $\vec{f} = \vec{f}_{(l)} + \vec{f}_{g(l)}$, where $\vec{f}_{(l)}$ is the SR value of the unbalanced relativistic force other than gravitational in the frame $S_4^{(l)}$, while the time coordinates in the two systems are taken the same, $\underline{x}^{\underline{0}} = \underline{t} = X^0 = T$. The (44) now can be replaced by

$$|\mathbf{a}| = \frac{1}{m\gamma}|\vec{f}_{(l)} + \vec{f}_{g(l)}|, \tag{81}$$

and according to (41), the general relativistic *inertial force* reads

$$\widetilde{\vec{f}}_{(in)} = -\frac{\sqrt{2}\underline{v}_c\vec{e}_f}{\gamma(1+\underline{\varrho}^2)^2}\,|f^{\alpha}_{(l)} - m\frac{\partial X^{\alpha}}{\partial\widetilde{x}^{\sigma}}\Gamma^{\sigma}_{\mu\nu}\frac{d\widetilde{x}^{\mu}}{ds}\frac{d\widetilde{x}^{\nu}}{ds}|. \tag{82}$$

As before, the two systems S_2 and $S_4^{(l)}$ can be chosen in such a way as the axis $e_{\underline{1}}$ of $S_{(2)}$ lies $(e_{\underline{1}} = \vec{e}_f)$ along the acting net force

$$\vec{f} = \vec{f}_{(l)} + \vec{f}_{g(l)}, \tag{83}$$

while the time coordinates in the two systems are taken the same

$$\underline{x}^{\underline{0}} = x^0 = X^0 = T. \tag{84}$$

Here $\vec{f}_{(l)}$ is the SR value of the unbalanced relativistic force other than gravitational and $\vec{f}_{g(l)}$ is the gravitational force given in the frame $S_4^{(l)}$. Despite of totally different and independent sources of gravitation and inertia, at $f^{\alpha}_{(l)} = 0$, the (82) establishes the independence of free-fall trajectories of the mass, internal composition and structure of bodies. This furnishes a justification for the introduction of the relativistic WPE. A remarkable feature is that, although the inertial force has a nature different than the gravitational force, nevertheless both are due to 2D $\widetilde{MS}_p \equiv \underline{V}_2$ and 4D-background space V_4, respectively.

6. The inertial effects in the background post Riemannian geometry

If the nonmetricity tensor

$$N_{\lambda\mu\nu} = -\mathcal{D}_{\lambda}\, g_{\mu\nu} \equiv -g_{\mu\nu\,;\lambda}$$

does not vanish, the general formula for the affine connection written in the space-time components is (Poplawski, 2009)

$$\Gamma^{\rho}{}_{\mu\,\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\,\nu} + K^{\rho}{}_{\mu\nu} - N^{\rho}{}_{\mu\nu} + \frac{1}{2}N_{(\mu}{}^{\rho}{}_{\nu)}, \tag{85}$$

where the metric alone determines the torsion-free Levi-Civita connection $\overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu}$,

$$K^{\rho}{}_{\mu\nu} := 2Q_{(\mu\nu)}{}^{\rho} + Q^{\rho}{}_{\mu\nu}$$

is the non-Riemann part - the affine *contortion tensor*. The torsion,

$$Q^{\rho}{}_{\mu\nu} = \frac{1}{2}T^{\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{[\mu\,\nu]},$$

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given with respect to a holonomic frame, $d\vartheta^\rho = 0$, is a third-rank tensor, antisymmetric in the first two indices, with 24 independent components. We now compute the relativistic inertial force for the motion of the matter, which is distributed over a small region in the U_4 space and consists of points with the coordinates x^μ, forming an extended body whose motion in the space, U_4, is represented by a world tube in space-time. Suppose the motion of the body as a whole is represented by an arbitrary timelike world line γ inside the world tube, which consists of points with the coordinates $\tilde{X}^\mu(\tau)$, where τ is the proper time on γ. Define

$$\delta x^\mu = x^\mu - \tilde{X}^\mu, \;\; \delta x^0 = 0, \;\; u^\mu = \tfrac{d\tilde{X}^\mu}{ds}. \tag{86}$$

The *Papapetrou equation of motion for the modified momentum* (Bergmann & Thompson, 1953, Møller, 1958, Papapetrou, 1974, Poplawski, 2009) is

$$\tfrac{\overset{\circ}{\mathcal{D}}\Theta^\nu}{\mathcal{D}s} = -\tfrac{1}{2}\, \overset{\circ}{R}{}^{\nu}{}_{\mu\sigma\rho}\, u^\mu J^{\sigma\rho} - \tfrac{1}{2} N_{\mu\rho\lambda} K^{\mu\rho\lambda:\nu}, \tag{87}$$

where $K^\mu_{\nu\lambda}$ is the contortion tensor,

$$\Theta^\nu = P^\nu + \tfrac{1}{u^0}\, \overset{\circ}{\Gamma}{}^{\nu}_{\mu\rho}\, (u^\mu J^{\rho 0} + N^{0\mu\rho}) - \tfrac{1}{2u^0} K_{\mu\rho}{}^{\nu} N^{\mu\rho 0} \tag{88}$$

is referred to as the *modified 4-momentum*,

$$P^\lambda = \int \tau^{\lambda 0}\, d\Omega,$$

is the ordinary 4-momentum, $d\Omega := dx^4$, and the following integrals are defined:

$$\begin{array}{l} M^{\mu\rho} = u^0 \int \tau^{\mu\rho}\, d\Omega, \quad M^{\mu\nu\rho} = -u^0 \int \delta x^\mu \tau^{\nu\rho}\, d\Omega, \quad N^{\mu\nu\rho} = u^0 \int s^{\mu\nu\rho}\, d\Omega, \\ J^{\mu\rho} = \int (\delta x^\mu \tau^{\rho 0} - \delta x^\rho \tau^{\mu 0} + s^{\mu\rho 0})\, d\Omega = \tfrac{1}{u^0}(-M^{\mu\rho 0} + M^{\rho\mu 0} + N^{\mu\rho 0}), \end{array} \tag{89}$$

where $\tau^{\mu\rho}$ is the energy-momentum tensor for particles, $s^{\mu\nu\rho}$ is the spin density. The quantity $J^{\mu\rho}$ is equal to

$$\int (\delta x^\mu \tau^{kl} - \delta x^\rho \tau^{\mu\lambda} + s^{\mu\rho\lambda})\, dS_\lambda,$$

taken for the volume hypersurface, is a tensor called the *total spin tensor*. The quantity $N^{\mu\nu\rho}$ is also a tensor. The relation $\delta x^0 = 0$ gives $M^{0\nu\rho} = 0$. It was assumed that the dimensions of the body are small, so integrals with two or more factors δx^μ multiplying $\tau^{\nu\rho}$ and integrals with one or more factors δx^μ multiplying $s^{\nu\rho\lambda}$ can be neglected. The *Papapetrou equations of motion for the spin* (Bergmann & Thompson, 1953, Møller, 1958, Papapetrou, 1974, Poplawski, 2009) is

$$\tfrac{\overset{\circ}{\mathcal{D}}}{\mathcal{D}s} J^{\lambda\nu} = u^\nu \Theta^\lambda - u^\lambda \Theta^\nu + K^\lambda_{\;\mu\rho} N^{\nu\mu\rho} + \tfrac{1}{2} K_{\mu\rho}{}^{\lambda} N^{\mu\nu\rho} - K^\nu_{\;\mu\rho} N^{\lambda\mu\rho} - \tfrac{1}{2} K_{\mu\rho}{}^{\nu} N^{\mu\rho\lambda}. \tag{90}$$

Computing from (87), in general, the relativistic inertial force, exerted on the extended spinning body moving in the RC space U_4, can be found to be

$$\begin{array}{l} \vec{f}_{(in)}(x) = -\tfrac{m\vec{a}_{abs}(x)}{\Omega^2(\underline{\varrho})\gamma} = -m\, \tfrac{\vec{e}_f}{\Omega^2(\underline{\varrho})\gamma} \left| \tfrac{1}{m} f^\alpha_{(l)} - \tfrac{\partial X^\alpha}{\partial x^\mu} \left[\overset{\circ}{\Gamma}{}^{\mu}_{\nu\lambda}\, u^\nu u^\lambda + \right.\right. \\ \left.\left. \tfrac{1}{u^0}\, \overset{\circ}{\Gamma}{}^{\mu}_{\nu\rho}\, (u^\nu J^{\rho 0} + N^{0\nu\rho}) - \tfrac{1}{2u^0} K_{\nu\rho}{}^{\mu} N^{\nu\rho 0} + \tfrac{1}{2}\, \overset{\circ}{R}{}^{\mu}{}_{\nu\sigma\rho}\, u^\nu J^{\sigma\rho} + \tfrac{1}{2} N_{\nu\rho\lambda} K^{\nu\rho\lambda:\mu} \right] \right|. \end{array} \tag{91}$$

7. Concluding remarks

In this section we briefly highlight a few key points. In the framework of $\widetilde{MS}_p$-TSG, we address the theory of a general deformation of MS_p induced by external force exerted on a particle. A coupling of supergravity with matter superfields no longer holds. Instead, the source of these fields is the deformation of the flat MS_p. Considering the simple spacetime deformation map, $\Omega(\underline{\varrho}) : \underline{M}_2 \to \underline{V}_2$ ($\Omega = \overset{\circ}{\Omega}$, $\breve{\Omega}^\mu{}_\nu \equiv \delta^\mu_\nu$), we have to write the rate, $\underline{\varrho}$, in terms of the Lorentz spinors $(\underline{\theta}, \underline{\bar{\theta}})$, and period of superoscillations (τ). A period of superoscillations, $\tau(\underline{s})$, can be defined as a function of proper time $(\underline{s})$ induced by the *world-deformation* tensor $\Omega(\underline{s})$. In this way we show that the occurrence of the *absolute* and *inertial* accelerations, and the *inertial* force, in turn, are obviously caused by a general deformation of flat MS_p. Whereas, the *relative* acceleration in 4D Minkowski space, M_4, (in Newton's terminology) (both magnitude and direction), to the

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contrary, has nothing to do with a deformation of $\underline{M}_2$ and, thus, it cannot produce the inertia effects. We calculate the relativistic inertial force in Minkowski, semi-Riemannian and post Riemannian spaces. Despite of totally different and independent sources of gravitation and inertia, the general inertial force establishes the independence of free-fall trajectories of the mass, internal composition and structure of bodies. This furnishes a justification for the introduction of the relativistic WPE. We discuss the inertia effects beyond the hypothesis of locality with special emphasis on deformation $\underline{M}_2 \longrightarrow \underline{V}_2^{(\varrho)}$, which essentially improves the standard framework. We derive the tetrad fields as a function of ϱ, describing corresponding *fictitious graviton*. Therewith the *fictitious gravitino*, $\psi_{\hat{m}}^{\ \alpha}(\varrho)$, will be arisen under infinitesimal transformations of local supersymmetry.

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Appendices

Appendix A The $\widetilde{MS}_p$-SG and $\widetilde{MS}_p$-TSG

Throughout we will use the 'two-in-one' notation of a theory MS$_p$-SUSY, implying that any tensor (W) or spinor (Θ) with indices marked by 'hat' denote

$$\begin{array}{l} W^{\hat{\mu}_1\cdots\hat{\mu}_m}_{\hat{\nu}_1\cdots\hat{\nu}_n} := W^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} \oplus W^{\underline{\mu}_1\cdots\underline{\mu}_m}_{\underline{\nu}_1\cdots\underline{\nu}_n}, \\ \Theta^{\hat{\alpha}} := \theta^{\alpha} \oplus \underline{\theta}^{\alpha}, \quad \bar{\Theta}_{\hat{\dot{\alpha}}} := \bar{\theta}_{\dot{\alpha}} \oplus \bar{\underline{\theta}}_{\dot{\alpha}}. \end{array} \tag{92}$$

This corresponds to the action of supercharge operators $Q \equiv$ (either q or $\underline{q}$), which is due to the fact that the framework of $\widetilde{MS}_p$-SG combines bosonic and fermionic states in V_4 and $\underline{V}_2$ on the same base rotating them into each other under the action of operators $(q, \underline{q})$. The α are all upper indices, while $\dot{\alpha}$ is a lower index.

A.1 The $\widetilde{MS}_p$-SG

A local extension of the MS$_p$-SUSY algebra leads to the gauge theory of *translations*. One might guess that the condition for the parameter $\partial_{\hat{\mu}}\epsilon = 0$ of a global MS$_p$-SUSY theory (Ter-Kazarian, 2023b, 2024a) should be relaxed for the accelerated particle motion, so that a global SUSY will be promoted to a local SUSY in which the parameter $\epsilon = \epsilon(X^{\hat{\mu}})$ depends explicitly on $X^{\hat{\mu}} = (\widetilde{x}^{\mu}, \widetilde{\underline{x}}^{\underline{\mu}}) \in V_4 \oplus \underline{V}_2$, where $\widetilde{x}^{\mu} \in V_4$ and $\widetilde{\underline{x}}^{\underline{\mu}} \in \widetilde{MS}_p (\equiv \underline{V}_2)$, with V_4 and $\underline{V}_2$ are the 4D and 2D semi-Riemannian spaces. This extension will address the *accelerated motion* and *inertia effects*.

A smooth embedding map, generalized for curved spaces, becomes

$$\widetilde{f}: \ \underline{V}_2 \longrightarrow V_4, \tag{93}$$

defined to be an immersion (the embedding which is a function that is a homeomorphism onto its image):

$$\widetilde{\underline{e}}_{\underline{0}} = \widetilde{e}_0, \quad \widetilde{\underline{x}}^{\underline{0}} = \widetilde{x}^0, \quad \widetilde{\underline{e}}_{\underline{1}} = \vec{\widetilde{n}}, \quad \widetilde{\underline{x}}^{\underline{1}} = |\vec{\widetilde{x}}|, \tag{94}$$

where $\vec{\widetilde{x}} = \widetilde{e}_i \widetilde{x}^i = \vec{\widetilde{n}}|\vec{\widetilde{x}}|$ $(i = 1, 2, 3)$ (the middle letters of the Latin alphabet $(i, j, ...)$ will be reserved for space indices). From embedding relations (94), we obtain the components of velocity of a particle

$$\begin{array}{l} \widetilde{\underline{v}}^{(\pm)} = \frac{d\underline{x}^{(\pm)}}{d\widetilde{\underline{x}}^{\underline{0}}} = \frac{1}{\sqrt{2}}(\widetilde{\underline{v}}^{\underline{0}} \pm \widetilde{\underline{v}}^{\underline{1}}), \\ \widetilde{\underline{v}}^{\underline{1}} = \frac{d\widetilde{\underline{x}}^{\underline{1}}}{d\widetilde{\underline{x}}^{\underline{0}}} = |\vec{\widetilde{v}}| = |\frac{d\vec{\widetilde{x}}}{d\widetilde{x}^0}|, \end{array} \tag{95}$$

so that

$$\begin{array}{l} \widetilde{\underline{u}} = \widetilde{\underline{e}}_{\underline{m}} \widetilde{\underline{v}}^{\underline{m}} = (\vec{\widetilde{\underline{v}}}_{\underline{0}}, \vec{\widetilde{\underline{v}}}_{\underline{1}}), \\ \vec{\widetilde{\underline{v}}}_{\underline{0}} = \widetilde{\underline{e}}_0 \widetilde{\underline{v}}^{\underline{0}}, \quad \vec{\widetilde{\underline{v}}}_{\underline{1}} = \widetilde{\underline{e}}_{\underline{1}} \widetilde{\underline{v}}^{\underline{1}} = \vec{\widetilde{n}}|\vec{\widetilde{v}}| = \vec{\widetilde{v}}, \end{array} \tag{96}$$

therefore, $\widetilde{\underline{u}} = (\vec{\widetilde{\underline{v}}}_{\underline{0}}, \vec{\widetilde{\underline{v}}}_{\underline{1}}) = \widetilde{u} = (\widetilde{e}_0, \vec{\widetilde{v}})$. Thence, the components of the acceleration vector, $\dot{a}^{\hat{\rho}} = (a^{\rho}, \underline{a}^{\underline{\rho}})$, satisfy the following embedding relations

$$\underline{a}^{\underline{0}} = a^0, \quad \underline{a}^{\underline{1}} = |\vec{a}|. \tag{97}$$

On the premises of (Ter-Kazarian, 2024a), we review the accelerated motion of a particle in a new perspective of local $\widetilde{MS}_p$-SUSY transformations that a *creation* of a particle in $\underline{V}_2$ means its transition from initial state defined on V_4 into intermediate state defined on $\underline{V}_2$, while an *annihilation* of a particle in $\underline{V}_2$ means vice versa. The same interpretation holds for the *creation* and *annihilation* processes in V_4. The net result of each atomic double transition of a particle $V_4 \rightleftharpoons \underline{V}_2$ to $\underline{V}_2$ and back is as if we had operated with *a local space-time translation* with acceleration, $\vec{a}$, on the original space V_4. Accordingly, the acceleration, $\underline{\vec{a}}$, holds in $\underline{V}_2$ at $\underline{V}_2 \rightleftharpoons V_4$. So, the accelerated motion of boson $A(\widetilde{x})$ in V_4 is a chain of its sequential transformations to the Weyl fermion $\underline{\chi}(\widetilde{\underline{x}})$ defined on $\underline{V}_2$ (accompanied with the auxiliary fields $\widetilde{\underline{F}}$) and back,

$$\rightarrow A(\widetilde{x}) \rightarrow \underline{\chi}^{(\underline{F})}(\widetilde{\underline{x}}) \rightarrow A(\widetilde{x}) \rightarrow \underline{\chi}^{(\underline{F})}(\widetilde{\underline{x}}) \rightarrow, \tag{98}$$

and the same interpretation holds for fermion $\chi(\widetilde{x})$.

The mathematical structure of a local theory of $\widetilde{MS}_p$-SUSY has much in common with those used in the geometrical framework of standard supergravity theories. Such a local SUSY can already be read off from the algebra of a global MS$_p$-SUSY (Ter-Kazarian, 2024a).

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A.2 The simple $(N=1)$ $\widetilde{MS}_p$ - SG without auxiliary fields, revisited

The generalized Poincaré superalgebra for the simple $(N=1)$ $\widetilde{MS}_p$-SG reads:

$$\begin{array}{l} [P_{\hat{a}}, P_{\hat{b}}] = 0, \quad [S_{\hat{a}\hat{b}}, P_{\hat{c}}] = (\eta_{\hat{a}\hat{c}} P_{\hat{b}} - \eta_{\hat{b}\hat{c}} P_{\hat{a}}), \\ [S_{\hat{a}\hat{b}}, S_{\hat{c}\hat{d}}] = i(\eta_{\hat{a}\hat{c}} S_{\hat{b}\hat{d}} - \eta_{\hat{b}\hat{c}} S_{\hat{a}\hat{d}} + \eta_{\hat{b}\hat{d}} S_{\hat{a}\hat{c}} qq - \eta_{\hat{a}\hat{d}} S_{\hat{b}\hat{c}}), \\ [S_{\hat{a}\hat{b}}, Q^{\alpha}] = \frac{1}{2}(\gamma_{\hat{a}\hat{b}})^{\alpha}_{\beta} Q^{\beta}, \\ [P_{\hat{a}}, Q^{\beta}] = 0, \quad [Q_{\alpha}, \bar{Q}_{\dot{\beta}}] = \frac{1}{2}(\gamma^{\hat{a}})_{\alpha\dot{\beta}} P_{\hat{a}}. \end{array} \tag{99}$$

with $(S_{\hat{a}\hat{b}})^{\hat{c}}_{\hat{d}} = i(\delta^{\hat{c}}_{\hat{a}}\eta_{\hat{b}\hat{d}} - \delta^{\hat{c}}_{\hat{b}}\eta_{\hat{a}\hat{d}})$ a given representation of the Lorentz generators. Using (99) and a general form for gauge transformations on B^A,

$$\delta B = \mathcal{D}\lambda = d\lambda + [B, \lambda], \tag{100}$$

with

$$\lambda = \rho^{\hat{a}} P_{\hat{a}} + \tfrac{1}{2}\kappa^{\hat{a}\hat{b}} S_{\hat{a}\hat{b}} + \bar{Q}\varepsilon, \tag{101}$$

we obtain that the $(e^{\hat{a}}, \omega^{\hat{a}\hat{b}}, \Psi)$ transform under Poincaré translations as

$$\delta e^{\hat{a}} = \mathcal{D}\rho^{\hat{a}}, \quad \delta\omega^{\hat{a}\hat{b}} = 0, \quad \delta\Psi = 0; \tag{102}$$

under Lorentz rotations as

$$\delta e^{\hat{a}} = \kappa^{\hat{a}}_{\hat{b}} \delta e^{\hat{b}}, \quad \delta\omega^{\hat{a}\hat{b}} = -\mathcal{D}\kappa^{\hat{a}\hat{b}}, \quad \delta\Psi = \tfrac{1}{4}\kappa^{\hat{a}\hat{b}}\gamma_{\hat{a}\hat{b}}\Psi; \tag{103}$$

and under supersymmetry transformation as

$$\delta e^{\hat{a}} = \tfrac{1}{2}\bar{\varepsilon}\gamma^{\hat{a}}\Psi, \quad \delta\omega^{\hat{a}\hat{b}} = 0, \quad \delta\Psi = \mathcal{D}\varepsilon. \tag{104}$$

In first-order formalism, the gauge fields $(e^{\hat{a}}, \omega^{\hat{a}\hat{b}}, \Psi)$, (with $\Psi = (\psi, \underline{\psi})$ a two-component Majorana spinor) are considered as an independent members of multiplet in the adjoint representation of the Poincaré supergroup of $D = 6$ $((3+1), (1+1))$ simple $(N=1)$ $\widetilde{MS}_p$-SG with the generators $(P_{\hat{a}}, S_{\hat{a}\hat{b}}, Q^{\alpha})$. Unless indicated otherwise, henceforth the world indices are kept implicit without ambiguity. The operators carry Lorentz indices not related to coordinate transformations. The Yang-Mills connection for the Poincare´ supergroup is given by

$$B = B^A T_A = e^{\hat{a}} P_{\hat{a}} + \tfrac{1}{2} i\omega^{\hat{a}\hat{b}} S_{\hat{a}\hat{b}} + \Psi\bar{Q}. \tag{105}$$

The field strength associated with connection B is defined as the Poincaré Lie superalgebra-valued curvature two-form R^A. Splitting the index A, and taking the $\Theta = \bar{\Theta} = 0$ component of R^A, we obtain

$$\begin{array}{l} R^{\ \hat{a}\hat{b}}(\omega) = d\omega^{\hat{a}\hat{b}} - \omega^{\hat{a}}{}_{\hat{c}}\omega^{\hat{c}\hat{d}}, \\ \tilde{T}^{\hat{a}} = T^{\hat{a}} - \frac{1}{2}\bar{\Psi}\gamma^{\hat{a}}\Psi, \quad \rho = \mathcal{D}\Psi, \end{array} \tag{106}$$

where $\gamma^{\hat{a}} = (\gamma^a, \sigma^{\underline{a}})$, $R^{\ \hat{a}\hat{b}}(\omega)$ is the Riemann curvature in terms of the spin connection $\omega^{\hat{a}\hat{b}}$, and the generalized Weyl lemma requires that the, so-called, supertorsion $\tilde{T}^{\hat{a}}$ be inserted. The solution $\omega(e)$ satisfies the tetrad postulate that the completely covariant derivative of the tetrad field vanishes, therefore $R^{\hat{a}\hat{b}}(\omega) = R(\omega)e^{\hat{a}}e^{\hat{b}}$.

The starting point of our approach is the action of a simple $\widetilde{MS}_p$-SG theory written in 'two in one'-notation (92), which is invariant under the local supersymmetry transformation (104), where the Poincaré superalgebra closes off shell without the need for any auxiliary fields:

$$\mathcal{L}_{MS-SG} = \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}} e^{\hat{a}} e^{\hat{b}} R^{\hat{c}\hat{d}}(\omega) + 4\bar{\Psi}\gamma_{\hat{5}} e^{\hat{a}}\gamma_{\hat{a}}\mathcal{D}\Psi. \tag{107}$$

This is the sum of bosonic and fermionic parts with the same spin connection, where $\gamma_{\hat{a}} = (\gamma_a \oplus \sigma_{\underline{a}})$, $\gamma_{\hat{5}} = (\gamma_5 \oplus \gamma_{\underline{5}})$, $\gamma_{\underline{5}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is given in the chiral or Weyl representations, i.e. in the irreducible 2-dimensional spinor representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, since two-component formalism works for a Weyl fermion. We can recast the generalized bosonic and fermionic actions given in (107), respectively, in the forms

$$\mathcal{L}^{(2)} = -\tfrac{1}{4}\sqrt{g}R(g, \Gamma) = -\tfrac{1}{4}eR(e, \omega), \tag{108}$$

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and

$$\mathcal{L}^{(3/2)} = 4\varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\bar{\Psi}_{\hat{\mu}}\gamma_{\hat{5}}\gamma_{\hat{\nu}}\mathcal{D}_{\hat{\rho}}\Psi_{\hat{\sigma}}. \tag{109}$$

The accelerated motion of a particle is described by the parameter $\epsilon = \epsilon(X^{\hat{\mu}})$ of local SUSY, which depends explicitly on $X^{\hat{\mu}} = (\widetilde{x}^{\mu}, \underline{\widetilde{x}}^{\underline{\mu}})$, where $\widetilde{x}^{\mu} \in V_4$ and $\underline{\widetilde{x}}^{\underline{\mu}} \in \underline{V}_2$. To be specific, let us focus for the motion on the simple case of a peculiar anticommuting spinors $(\underline{\xi}(\underline{x}), \bar{\underline{\xi}}(\underline{x}))$ and $(\xi(x), \bar{\xi}(x))$ defined as

$$\begin{array}{ll} \underline{\xi}^{\alpha}(\underline{x}) = i\,\frac{\underline{\tau}(\underline{x})}{2}\,\underline{\theta}^{\alpha}, & \bar{\underline{\xi}}_{\dot{\alpha}}(\underline{x}) = -i\,\frac{\underline{\tau}^*(\underline{x})}{2}\,\bar{\underline{\theta}}_{\dot{\alpha}}, \\ \xi^{\alpha}(x) = i\frac{\tau(x)}{2}\theta^{\alpha}, & \bar{\xi}_{\dot{\alpha}}(x) = -i\frac{\tau^*(x)}{2}\bar{\theta}_{\dot{\alpha}}. \end{array} \tag{110}$$

Here the real parameter $\tau(x) = \tau^*(x) = \underline{\tau}(\underline{x}) = \underline{\tau}^*(\underline{x})$ can physically be interpreted as the *atomic duration time* of double transition of a particle $V_4 \rightleftharpoons \underline{V}_2$, i.e. the period of superoscillations. In this case, the *atomic displacement* caused by double transition reads

$$\Delta\underline{\widetilde{x}}_{(a)} = \underline{\widetilde{e}}_{\underline{m}}\Delta\underline{\widetilde{x}}^{\underline{m}}_{(a)} = \underline{\widetilde{u}}\tau(\underline{\widetilde{x}}), \tag{111}$$

where the components $\Delta\underline{\widetilde{x}}^{\underline{m}}_{(a)}$ are written

$$\Delta\underline{\widetilde{x}}^{\underline{m}}_{(a)} = \underline{\widetilde{v}}^{\underline{m}}\,\tau(\underline{\widetilde{x}}) = i\underline{\theta}\,\sigma^{\underline{m}}\,\bar{\underline{\xi}}(\underline{\widetilde{x}}) - i\underline{\xi}(\underline{\widetilde{x}})\,\sigma^{\underline{m}}\,\bar{\underline{\theta}}. \tag{112}$$

In Van der Warden notations for the Weyl two-component formalism, we have

$$\underline{v}^2 = 2v^{(+)}v^{(-)} = (\underline{v}^{\underline{0}})^2 - (\underline{v}^{\underline{1}})^2 = 4(\underline{\theta}_1\,\bar{\underline{\theta}}_1\underline{\theta}_2\,\bar{\underline{\theta}}_2)\frac{d\tau}{d\underline{x}^{(+)}}\frac{d\tau}{d\underline{x}^{(-)}} = 1, \tag{113}$$

provided,

$$\begin{array}{l} a^{(+)} = \sqrt{2}v_c^{(+)}\frac{d^2\tau}{d\underline{x}^{(+)2}}, \\ a^{(-)} = \sqrt{2}v_c^{(-)}\frac{d^2\tau}{d\underline{x}^{(-)2}}, \\ \underline{a} = \sqrt{2}(a^{(+)}a^{(-)})^{1/2} = 2(\underline{\theta}_1\,\bar{\underline{\theta}}_1\underline{\theta}_2\,\bar{\underline{\theta}}_2)^{1/2}\frac{d^2\tau}{d\underline{s}^2}, \end{array} \tag{114}$$

with $v_c^{(+)} = \sqrt{2}(\underline{\theta}_1\,\bar{\underline{\theta}}_1)$ and $v_c^{(-)} = \sqrt{2}(\underline{\theta}_2\,\bar{\underline{\theta}}_2)$. The acceleration will generally remain a measure of the velocity variation over proper time ($\underline{s}$). The (114) gives

$$\begin{array}{l} v^{(+)} = v_c^{(+)}\left(\frac{d\tau}{d\underline{x}^{(+)}} + 1\right), \\ v^{(-)} = v_c^{(-)}\left(\frac{d\tau}{d\underline{x}^{(-)}} + 1\right), \\ \underline{v} = \sqrt{2}(v^{(+)}v^{(-)})^{1/2} = 2(\underline{\theta}_1\,\bar{\underline{\theta}}_1\underline{\theta}_2\,\bar{\underline{\theta}}_2)^{1/2}\left(\frac{d\tau}{d\underline{s}} + 1\right), \end{array} \tag{115}$$

where $d\underline{s}^2 = d\underline{x}^{(+)}d\underline{x}^{(-)}$. The spinors $\theta(\underline{\theta}, \bar{\underline{\theta}})$ and $\bar{\theta}(\underline{\theta}, \bar{\underline{\theta}})$ satisfy the embedding map (94), namely $\Delta\underline{\widetilde{x}}^{\underline{0}} = \Delta\widetilde{x}^0$ and $(\Delta\underline{\widetilde{x}}^{\underline{1}})^2 = (\Delta\vec{\widetilde{x}})^2$, so we obtain

$$\begin{array}{l} \underline{\theta}\,\sigma^0\,\bar{\underline{\xi}} - \underline{\xi}\,\sigma^0\,\bar{\underline{\theta}} = \theta\,\sigma^0\,\bar{\xi} - \xi\,\sigma^0\,\bar{\theta}, \\ (\underline{\theta}\,\sigma^3\,\bar{\underline{\xi}} - \underline{\xi}\,\sigma^3\,\bar{\underline{\theta}})^2 = (\theta\,\vec{\sigma}\,\bar{\xi} - \xi\,\vec{\sigma}\,\bar{\theta})^2. \end{array} \tag{116}$$

Denote

$$\begin{array}{l} \underline{v}^{\underline{0}}_{(c)} = \frac{1}{\sqrt{2}}\left(v_c^{(+)} + v_c^{(-)}\right) = (\underline{\theta}\,\bar{\underline{\theta}}), \\ \underline{v}^{\underline{1}}_{(c)} = \frac{1}{\sqrt{2}}\left(v_c^{(+)} - v_c^{(-)}\right) = (\underline{\theta}_1\,\bar{\underline{\theta}}_1 - \underline{\theta}_2\,\bar{\underline{\theta}}_2), \end{array} \tag{117}$$

then

$$\begin{array}{l} \theta_1(\underline{\theta}, \bar{\underline{\theta}}) = \frac{1}{2}\left[\left(\underline{v}^{\underline{0}}_{(c)} + \sqrt{\frac{2}{3}}\underline{v}^{\underline{1}}_{(c)}\right)^{1/2} + \left(\underline{v}^{\underline{0}}_{(c)} - \sqrt{\frac{2}{3}}\underline{v}^{\underline{1}}_{(c)}\right)^{1/2}\right], \\ \theta_2(\underline{\theta}, \bar{\underline{\theta}}) = \frac{1}{2}\left[\left(\underline{v}^{\underline{0}}_{(c)} + \sqrt{\frac{2}{3}}\underline{v}^{\underline{1}}_{(c)}\right)^{1/2} - \left(\underline{v}^{\underline{0}}_{(c)} - \sqrt{\frac{2}{3}}\underline{v}^{\underline{1}}_{(c)}\right)^{1/2}\right]. \end{array} \tag{118}$$

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A.3 The $\widetilde{MS}_p$-TSG with the translation group

In Teleparallel Gravity, the spin connection represents only *inertial effects*, but not gravitation at all. All quantities related to Teleparallel Gravity will be denoted with an over 'dot'. Teleparallel Gravity is a gauge theory for the *translation* group (de Andrade & Pereira, 1997). The $\widetilde{MS}_p$-TSG theory, therefore, has the gauge *translation* group in tangent bundle. Namely, at each point p of coordinates X of the base space $(V_4 \oplus \underline{V}_2)$, there is attached a Minkowski tangent-space (the fiber) $T_p(V_4 \oplus \underline{V}_2) = T_{X^{\hat{\mu}}}(V_4 \oplus \underline{V}_2)$, on which the point dependent gauge transformations,

$$X'^{\hat{a}} = X^{\hat{a}} + \varepsilon^{\hat{a}}(X), \tag{119}$$

take place. Under an infinitesimal tangent space translation, it transforms according to

$$\delta\Phi(X^{\hat{a}}(X^{\hat{\mu}})) = -\varepsilon^{\hat{a}}\partial_{\hat{a}}\Phi(X^{\hat{a}}(X^{\hat{\mu}})). \tag{120}$$

The generators of this group satisfy the Lie algebra $[P_{\hat{a}}, P_{\hat{b}}] = 0$. In order to recover the covariance, it is necessary to introduce a 1-form of the Yang–Mills connection assuming values in the Lie algebra of the translation group:

$$B = e^{\hat{a}}P_{\hat{a}}, \tag{121}$$

with gauge field $e^{\hat{a}}$. Introducing the covariant derivative

$$\dot{\mathcal{D}}_{\hat{\mu}}X^{\hat{a}} = \partial_{\hat{\mu}}X^{\hat{a}} + \dot{\omega}^{\hat{a}}{}_{\hat{b}\hat{\mu}}X^{\hat{b}}, \tag{122}$$

the tetrad, which is invariant under translations, becomes

$$\dot{e}^{\hat{a}}{}_{\hat{\mu}} = \dot{\mathcal{D}}_{\hat{\mu}}X^{\hat{a}} + \dot{\omega}^{\hat{a}}{}_{\hat{b}\hat{\mu}}. \tag{123}$$

In this new class of frames, the gauge field transforms according to $\delta e^{\hat{a}}{}_{\hat{\mu}} = -\dot{\mathcal{D}}_{\hat{\mu}}\varepsilon^{\hat{a}}$. Thus the covariant derivative, $\dot{\mathcal{D}} = d + B$, with Yang–Mills connection reads

$$\dot{\mathcal{D}}_{\hat{\mu}} = (\delta^{\hat{a}}_{\hat{\mu}} + e_{\hat{\mu}}{}^{\hat{a}})\partial_{\hat{a}} = (\partial_{\hat{\mu}}X^{\hat{a}} + e_{\hat{\mu}}{}^{\hat{a}})\partial_{\hat{a}} = \dot{e}_{\hat{\mu}}{}^{\hat{a}}\partial_{\hat{a}}. \tag{124}$$

The curvature of the Weitzenböck connection

$$\dot{\Gamma}^{\hat{\rho}}{}_{\hat{\nu}\hat{\mu}} = \dot{e}_{\hat{a}}{}^{\hat{\rho}}\dot{\mathcal{D}}_{\hat{\mu}}\dot{e}^{\hat{a}}{}_{\hat{\nu}}, \tag{125}$$

vanishes identically, while for a tetrad $\dot{e}^{\hat{a}}$ with $e^{\hat{a}}{}_{\hat{\mu}} \neq \dot{\mathcal{D}}_{\hat{\mu}}\varepsilon^{\hat{a}}$, the torsion 2-form - the field strength (here we re-instate the factor $\wedge$),

$$\dot{T}^{\hat{a}} = d\dot{e}^{\hat{a}} = \frac{1}{2}\dot{T}^{\hat{a}}{}_{\hat{b}\hat{c}}\dot{e}^{\hat{b}} \wedge \dot{e}^{\hat{c}} = \dot{K}_{\hat{c}}{}^{\hat{a}} \wedge \dot{e}^{\hat{c}}, \tag{126}$$

is non-vanishing:

$$\dot{T}^{\hat{a}}{}_{\hat{\mu}\hat{\nu}} = \dot{\mathcal{D}}_{\hat{\mu}}\dot{e}^{\hat{a}}{}_{\hat{\nu}} - \dot{\mathcal{D}}_{\hat{\nu}}\dot{e}^{\hat{a}}{}_{\hat{\mu}} = \dot{\Gamma}^{\hat{a}}{}_{[\hat{\mu}\hat{\nu}]} = \dot{\mathcal{D}}_{\hat{\mu}}e^{\hat{a}}{}_{\hat{\nu}} - \dot{\mathcal{D}}_{\hat{\nu}}e^{\hat{a}}{}_{\hat{\mu}} \neq 0. \tag{127}$$

Here $\dot{K}^{\hat{a}\hat{b}}$ is the contorsion tensor, and we also taken into account the vanishing torsion, $[\dot{\mathcal{D}}_{\hat{\mu}}, \dot{\mathcal{D}}_{\hat{\nu}}]X^{\hat{a}} = 0$, of *inertial* tetrad, $\dot{e}^{\hat{a}}{}_{\hat{\mu}} = \dot{\mathcal{D}}_{\hat{\mu}}X^{\hat{a}}$. Hence

$$[\dot{e}_{\hat{\mu}}, \dot{e}_{\hat{\nu}}] = \dot{T}_{\hat{\mu}\hat{\nu}} = \dot{T}^{\hat{a}}{}_{\hat{\mu}\hat{\nu}}P_{\hat{a}}. \tag{128}$$

Due to the soldered character of the tangent bundle, torsion presents also the anholonomy of the translational covariant derivative:

$$[\dot{e}_{\hat{\mu}}, \dot{e}_{\hat{\nu}}] = \dot{T}_{\hat{\mu}\hat{\nu}} = \dot{T}^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}}P_{\hat{\rho}}. \tag{129}$$

The gauge invariance of the tetrad provides torsion invariance under gauge transformations. As a gauge theory for the translation group, the action of the $\widetilde{MS}_p$-TSG theory can be recast in the form (see also (Salgado et al., 2005))

$$\begin{aligned}\dot{\mathcal{L}}_{MS-TSG} &= \frac{1}{4}tr\left(\hat{\dot{T}} \wedge \star\hat{\dot{T}}\right) - 4\bar{\Psi}\gamma_{\hat{5}}\gamma_{\hat{d}}\mathcal{D}\Psi\dot{e}^{\hat{d}} \\ &= \frac{1}{4}\eta_{\hat{a}\hat{b}}\dot{T}^{\hat{a}} \wedge \star\dot{T}^{\hat{b}} - 4\bar{\Psi}\gamma_{\hat{5}}\gamma_{\hat{d}}\mathcal{D}\Psi\dot{e}^{\hat{d}},\end{aligned} \tag{130}$$

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where (we re-instate the factor $\wedge$) the torsion 2-form reads

$$\dot{\hat{T}} = \frac{1}{2}\dot{T}^{\hat{a}}{}_{\hat{\mu}\hat{\nu}} P_{\hat{a}} dX^{\hat{\mu}} \wedge dX^{\hat{\nu}}, \tag{131}$$

and

$$\star\dot{\hat{T}} = \frac{1}{2}\left(\star\dot{T}^{\hat{a}}{}_{\hat{\rho}\hat{\sigma}}\right) P_{\hat{a}} dX^{\hat{\rho}} \wedge dX^{\hat{\sigma}}. \tag{132}$$

Here $\star$ denotes the Hodge dual. Defining the tensor of superpotential

$$\dot{S}_{\hat{a}}{}^{\hat{\rho}\hat{\sigma}} = -\dot{S}_{\hat{a}}{}^{\hat{\sigma}\hat{\rho}} := \dot{K}^{\hat{\rho}\hat{\sigma}}{}_{\hat{a}} - \dot{e}_{\hat{a}}{}^{\hat{\sigma}}\dot{T}^{\hat{c}\hat{\rho}}{}_{\hat{c}} + \dot{e}_{\hat{a}}{}^{\hat{\rho}}\dot{T}^{\hat{c}\hat{\sigma}}{}_{\hat{c}}, \tag{133}$$

the dual torsion can be rewritten in the form

$$\star\dot{T}^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}} = \frac{\dot{e}}{2}\varepsilon_{\hat{\mu}\hat{\nu}\hat{\lambda}\hat{\sigma}}\dot{S}^{\hat{\rho}\hat{\lambda}\hat{\sigma}}, \tag{134}$$

and making use of the identity $\dot{T}^{\hat{\mu}}{}_{\hat{\mu}\hat{\rho}} = \dot{K}^{\hat{\mu}}{}_{\hat{\rho}\hat{\mu}}$, the action (130) becomes

$$\dot{\mathcal{L}}_{MS-TSG} = -\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}}\dot{K}^{\hat{a}\hat{b}}\dot{T}^{\hat{c}}\dot{e}^{\hat{d}} - 4\bar{\Psi}\gamma_{\hat{5}}\gamma_{\hat{d}}\mathcal{D}\Psi\dot{e}^{\hat{d}} + \text{surface term}. \tag{135}$$

This action is invariant under local translations, under local super symmetry transformations and by construction is invariant under local Lorentz rotations and under diffeomorphisms (see Salgado et al. (2003, 2005), Stelle & West (1980)). In other words, this action is invariant under the Poincaré supergroup and under diffeomorphisms. Consequently we proved the equivalence of the Teleparallel Gravity action $\dot{\mathcal{L}}^{(2)}$ with Hilbert action $\mathcal{L}^{(2)}$:

$$\dot{\mathcal{L}}^{(2)} = \mathcal{L}^{(2)} + \text{surface term}. \tag{136}$$

The equation of motion in the X-space is written as

$$\frac{du^{\hat{a}}}{ds} = \left(\dot{K}^{\hat{a}}{}_{\hat{b}\hat{\rho}} - \dot{\omega}^{\hat{a}}{}_{\hat{b}\hat{\rho}}\right) u^{\hat{b}}u^{\hat{\rho}}. \tag{137}$$

This equation can be rewritten in a purely spacetime form

$$\frac{du^{\hat{\rho}}}{ds} = \left(\dot{K}^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}} - \dot{\Gamma}^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}}\right) u^{\hat{\mu}}u^{\hat{\nu}}. \tag{138}$$

The corresponding acceleration cannot be given a covariant meaning without a connection, while each different connection $\Gamma^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}}$ will deffne a different acceleration. The Weitzenböck connection, which defines the Fock-Ivanenko derivative $\dot{\mathcal{D}}_{\hat{\mu}}$ written in terms of covariant derivative $\dot{\nabla}_{\hat{\mu}}$:

$$\dot{\mathcal{D}}_{\hat{\mu}}\Phi^{\hat{a}} = \dot{e}^{\hat{a}}{}_{\hat{\rho}}\dot{\nabla}_{\hat{\mu}}\Phi^{\hat{\rho}}, \tag{139}$$

will define the acceleration too

$$\begin{aligned} \dot{a}^{\hat{\rho}} &= \frac{\dot{\nabla}u^{\hat{\rho}}}{\dot{\nabla}s} = u^{\hat{\nu}}\dot{\nabla}_{\hat{\nu}}u^{\hat{\rho}} = \frac{du^{\hat{\rho}}}{ds} + \dot{\Gamma}^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}}u^{\hat{\mu}}u^{\hat{\nu}} \\ &= \dot{K}^{\hat{\rho}}{}_{\hat{\mu}\hat{\nu}}u^{\hat{\mu}}u^{\hat{\nu}} = \dot{T}^{\hat{\rho}}{}_{\hat{\nu}\hat{\mu}}u^{\hat{\mu}}u^{\hat{\nu}}. \end{aligned} \tag{140}$$

This is a force equation, with torsion (or contortion) playing the role of force. The dynamical aspects of particle mechanics involve derivatives with respect to proper time along the particle worldline, which is the line element written in frame:

$$ds^2 = \eta_{\hat{a}\hat{b}}\dot{e}^{\hat{a}}\dot{e}^{\hat{b}} = \eta_{\hat{a}\hat{b}}\dot{e}^{\hat{a}}{}_{\hat{\mu}}\dot{e}^{\hat{b}}{}_{\hat{\mu}}dX^{\hat{\mu}}dX^{\hat{\nu}} \equiv \eta_{\hat{\mu}\hat{\nu}}dX^{\hat{\mu}}dX^{\hat{\nu}}. \tag{141}$$

A worldline C of a particle, parametrized by proper time as $C(s) = X^{\hat{\mu}}(s)$, will have as six-velocity the vector of components $u^{\hat{\mu}} = dX^{\hat{\mu}}/ds$ and $u^{\hat{a}} = \dot{e}^{\hat{a}}_{\hat{\mu}}u^{\hat{\mu}}$, which are the particle velocity along this curve respectively in the holonomic and anholonomic bases in the X-space. The proper time can be written in the form $ds = u_{\hat{\mu}}dX^{\hat{\mu}} = u_{\hat{a}}\dot{e}^{\hat{a}}$. To transform the tetrad field into a reference frame in X-space with an observer attached to it, we may "attach" $\dot{e}_{\hat{0}}$ to the observer by identifying $u = \dot{e}_{\hat{0}} = \frac{d}{ds}$ with components $u^{\hat{\mu}} = \dot{e}_{\hat{0}}{}^{\hat{\mu}}$, such that $\dot{e}_{\hat{0}}$ will be the observer velocity. The Weitzenböck connection, $\dot{\Gamma}$, will attribute to the observer an acceleration

$$\dot{a}^{\hat{a}}_{(f,\dot{\Gamma})} = \dot{\omega}^{\hat{a}}{}_{\hat{0}\hat{0}} + \dot{K}^{\hat{a}}{}_{\hat{0}\hat{0}}, \tag{142}$$

seen by that very observer. Whereas,

$$\dot{\omega}^{\hat{a}}{}_{\hat{b}\hat{c}} = \dot{e}^{\hat{a}}{}_{\hat{\mu}}\dot{\nabla}_{\dot{e}_{\hat{c}}}, \dot{e}_{\hat{b}}{}^{\hat{\mu}}, \tag{143}$$

which literarily means the covariant derivative of $\dot{e}_{\hat{b}}$ along $\dot{e}_{\hat{c}}$, projected along $\dot{e}_{\hat{a}}$. As $\dot{a}^{\hat{\rho}}$ (140) is orthogonal to $u^{\hat{\rho}}$, its vanishing means that the $u^{\hat{\rho}}$ keeps parallel to itself along the worldline.

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