Radiation and Scattering of Massive Photons

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Abstract

With the aid of a modified Planck's law for massive photons, it is shown that the ratio of the mean value of the photon mass equivalent to its rest (invariant) mass tends to be one with a decrease in temperature. A modified Stefan - Boltzmann law is obtained at different temperature regimes, as well as the Wien's displacement law. At high temperatures the modified Planck' s law approaches the standard Planck' s law. It is also shown that the cross-section of the Thomson scattering slightly increases opacity of the scattering medium. The Compton shift in frequency for a massive photon appears to be frequency-dependent and slightly less than its value for a massless photon, except in the case of forward scattering when no change in frequency takes place. Astrophysical aspects of the massive photon hypothesis are discussed with regard to standard stellar models, early stages of the Universe, and the Breit-Weeler process, as well as active galactic nuclei. Estimates of the spreading time of the wave packet of the massive photon show that for frequencies $\nu \geq 4.052 \times 10^9 Hz (\lambda \leq 7.4\, cm)$ it exceeds the age of the Universe.

Keywords: *masive photon, optical dispersion in vacuum; modified Planck's law, Thomson scattering, Compton scattering, spreading time of the wave packet*

1. Introduction

The hypothesis that the photon has a nonzero rest mass has been the matter of numerous discussions over the past decades. Though this topic falls out of the traditional field of physical studies, it is of interest because of its fundamental impact on many aspects of physics and astrophysics. The mathematical basis for discussing the idea of photons with nonzero mass is the Proca equation (Poenaru, 2006, Tu et al., 2005) for massive vector bosons with spin 1. In the covariant form, it is a generalization of Maxwell's equations in the case of a massive electromagnetic field. The presence of a term with the mass of the photon in the Proca equation violates the condition of gauge invariance in electrodynamics and leads to a series of consequences, of which we note the effect of optical dispersion in vacuum - the hypothesis of an intrinsic dispersion of light, associated with the presumable dependence of its speed on the frequency of radiation. The effect of dispersion in vacuum is typical for de Broglie waves, which formally makes them akin to a massive electromagnetic field. Unlike the latter, de Broglie waves do not carry any physical interaction but are only a form of describing the behavior of particles within the framework of wave-particle duality. However, de Broglie wavelength of a photon coincides with the wavelength of an electromagnetic radiation associated with the photon (Blokhintsev, 1976). This gives a reason to at least formally consider it as a concentrated electromagnetic wave packet consisting of quasi-monochromatic waves. Bass & Schredinger (1955), who brought attention to the possibility of existence of "weighted" photons, noted that the presence of the non-zero mass of the photon leads to the third, longitudinal, polarization of radiation instead of two transverse polarizations in the normal, massless one. De Brogle & Vigier (1972) pointed out to a physical experiment with laser beams, resulted in the assumption on the possibility of existence of massive photons with the rest mass not exceeding $10^{-48})g$. Measurements of the relative variation in the speed of light$\delta c/c$ (the ratio of the velocity variation (δc) to its invariant value $c = 2.99792458 \times 10^8 m/s$, which used in the Lorentz transformations), produced by different authors at different radio frequencies under terrestrial conditions (see Tu et al., 2005, and references therein), confirmed the constancy of the speed of light with an accuracy of $\sim 10^{-7} - 10^{-4}$ and its fractional uncertainty $\sim 10^{-9}$ (Evenson & et al., 1972, Sullivan, 2009). Studies

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of active processes in distant astrophysical objects Shaefer (1999) show a relative variation in the speed of light, starting from orders of magnitude $\sim\ 10^{-21} - 10^{-12}$, resulting from the estimate of the difference in the time of arrival of signals at different frequencies (time dispersion effect). The existence of the nonzero rest mass of the photon is assumed in some models of quantum gravity Biller & et al. (1999). Obviously, in this case we are dealing with a very weak physical effect, the direct detection of which is currently still below the experimental threshold as decades ago (Goldhaber & Nieto, 1976). This circumstance, which has no effect on physics in everyday reality, may be more significant in extreme astrophysical conditions. If the hypothesis about nonzero mass of the photon has a right to exist, then one of the questions that naturally arises from it is how it may affect the nature of blackbody radiation. This problem was first considered by Torres-Hernandez (1985)by making use of a partition function Feynman (2018). He referred to the remark in the work of Bass & Schredinger (1955) that the contribution of longitudinally polarized photons in achieving of thermodynamic equilibrium in the black-body cavity can be neglected (the probability of the transition of "transverse" photons into "longitudinal" photons is extremely small, as shown by Kobzarev & Okun (1968). In this paper we are going to uncover some mathematical details just briefly mentioned by Torres-Hernandez (1985), but seem to be important for completeness of theoretical consideration of the topic. We show that at low temperatures the rest mass of the photon approaches the mean mass equivalent (Saiyan, 2023)of the Planck' photons.In addition to this we derive Stefan-Boltzmann law for high and low temperatures in connection with the radiation pressure. We derive Wien's displacement law regarding massive photons and discuss the effect of nonzero mass of the photons on Thomson and Compton scattering. Some astrophysical aspects, related to the results presented here, are discussed in Section 3 alongside with spreading time of wave packets of photons of different frequencies.

2. Modified Planck's law (MPL)

The formula, obtained by Torres-Hernandez (1985) for the spectral energy density of radiation, can be written in the form

$$\rho(\nu, m_\gamma, T) = \rho(\nu, T)\sqrt{1 - (\frac{m_\gamma c^2}{h\nu})^2} \tag{1}$$

which we call here and thereafter the modified Planck's law (MPL). Here ν is the radiation frequency, m_γ -the rest mass of the photon, T is the temperature of the blackbody radiation and

$$\rho(\nu, T) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{(\frac{h\nu}{kT})} - 1} \tag{2}$$

is the standard Planck spectral density of radiation, $h = 6.626 \times 10^{-34} Js$ is the Planck constant and $k = 1.38 \times 10^{-23}\frac{J}{K}$ is the Boltzmann constant. This topic was also discussed by Pardy (2018) and Nyambuya (2017), but without reference to the work of Torres-Hernandez (1985), which seems to us physically more consistent. From expression (1) it follows that at $m_\gamma = \frac{h\nu}{c^2} = 7.362 \times 10^{-51}\nu(kg)$ Planck's law turns to zero. This allows us to introduce a purely formal definition of the "rest frequency" of the photon

$$\nu_0 = \frac{m_\gamma c^2}{h} = 1.358 \times 10^{50} m_\gamma \quad (Hz) \tag{3}$$

if the mass is determined independently. The estimates of the rest mass are related to the measurement methodology and the frequency range used for this Butto & Tikva (2022) and Saiyan (2023), although the value of m_γ must be constant by its very meaning. Entering a dimensionless variable in (1) $x = \frac{h\nu}{kT}$, we write:

$$\frac{m_\gamma c^2}{h\nu} = \frac{\beta}{x}, \quad \beta = \frac{m_\gamma c^2}{kT} = 6.522 \times 10^{39}\frac{m_\gamma}{T}. \tag{4}$$

At high temperatures $m_\gamma c^2 \ll kT(\beta \ll 1)$, the MPL tends to be the standard Planck's law. As can be seen in (2), in this case

$$m_\gamma \ll 1.533 \times 10^{-40} T(kg). \tag{5}$$

At low temperatures $\beta \gg 1, x \gg 1$, both functions asymptotically tend to zero.

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2.1. Derivation of MPL

Here we will give a derivation of MPL that is somewhat different from the one given by Torres-Hernandez (1985) but having more details and leading to the same result. We start with the dispersion equation which can be obtained from the Proca equation (Tu et al., 2005):

$$k^2 = \frac{\omega^2}{c^2} - (\frac{m_\gamma c}{\hbar})^2 = 4\frac{\pi^2}{c^2}(\nu^2 - \nu_0^2), \quad \hbar = \frac{h}{2\pi}, \tag{6}$$

where k is the modulus of the wave vector ω is the cyclic frequency $\omega = 2\pi\nu$ and which we will rewrite in the form

$$k = 2\frac{\pi}{c}\sqrt{\nu^2 - \nu_0^2}. \tag{7}$$

Then the density of the number of oscillatory modes in the interval of wave numbers $(k, k+dk)$ in volume V of a black-body radiating cavity within which thermodynamic equilibrium has been achieved is equal to $dN(k) = \frac{k^2}{2\pi^2}Vdk$, or, in terms of frequencies, after differentiating equation (6), and taking into account two states of polarization of the photon,

$$dN(\nu) = 8\frac{\pi V}{c^3}\nu^2\sqrt{1 - (\frac{m_\gamma c^2}{h\nu})^2} \tag{8}$$

For the density of the number of states with radiation energy in the frequency interval $(\nu, \nu + d\nu)$, we have:

$$h\nu\frac{dN(\nu)}{\nu} = \frac{8\pi h\nu^3}{c^3}\sqrt{1 - (\frac{m_\gamma c^2}{h\nu})^2}\,d\nu \tag{9}$$

Because the mean occupation number of each energy state is given by the Planck distribution $\langle n \rangle = \frac{1}{e^{(\frac{h\nu}{kT})} - 1}$ (Feynman, 2018) for MPL we find

$$\rho(\nu, m_\gamma, T) = (\frac{8\pi h\nu^3}{c^3})\sqrt{1 - (\frac{m\gamma c^2}{h\nu})^2}\frac{1}{e^{(\frac{h\nu}{kT})} - 1}, \tag{10}$$

which coincides with (1). In fact, MPL differs from the standard Planck's law only in the presence of a radical expression. The average energy of a Planck photon will then be determined by the ratio of the total energy density in the radiation spectrum to the density of the number of photons, i.e., the expression

$$\langle\epsilon\rangle = kT\frac{\int_{h\nu_0}^{\infty}\rho(\nu, m_\gamma, T)d\nu}{\int_{h\nu_0}^{\infty}\frac{\rho(\nu, m_\gamma, T)\nu}{d}\nu} \quad \epsilon = h\nu. \tag{11}$$

Consequently, the average mass of the Planck photon (the average mass equivalent) is $\langle m \rangle = \frac{\langle\epsilon\rangle}{c^2}$. As follows from (11), for the standard Planck's law with $(\nu_0 = 0)$ $\langle\epsilon\rangle = 2.701kT$. Using expression (4), it is possible to obtain the temperature-dependent ratio of the average mass of the photon to its rest mass. To do this, let us rewrite MPL (10) in the form:

$$\rho(x, m_\gamma, T) = (\frac{8\pi}{(hc)^3}(kT)^4\frac{x^3}{e^x - 1}\sqrt{1 - (\frac{\beta}{x})^2}, \quad x \geq \beta. \tag{12}$$

The graph of the function (normalized on the constant factor) is shown on the Fig 1. below for different values of the parameter β. The graph with $\beta 0$ corresponds to the standard Planck's law and depends only on temperature. The graphs with $\beta > 0$ are dependent on temperature and the rest mass of the photon. Unlike the standard Planck's law, which turns into zero for $x0$, MPL turns into zero for $x = \beta$. Increase in β is equivalent to decrease in temperature. For $x \ll 1$ we get an analogue of the Rayleigh-Jeans law

$$\rho(x, m_\gamma, T) = \frac{8\pi}{(hc)^3}(kT)^4x^2\sqrt{1 - (\frac{\beta}{x})^2} \tag{13}$$

The condition $x \gg 1$ leads to Wien's radiation law:

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$$\rho(x, m_\gamma, T) = \frac{8\pi}{(hc)^3}(kT)^4 x^2 \sqrt{1 - (\frac{\beta}{x})^2}. \tag{14}$$

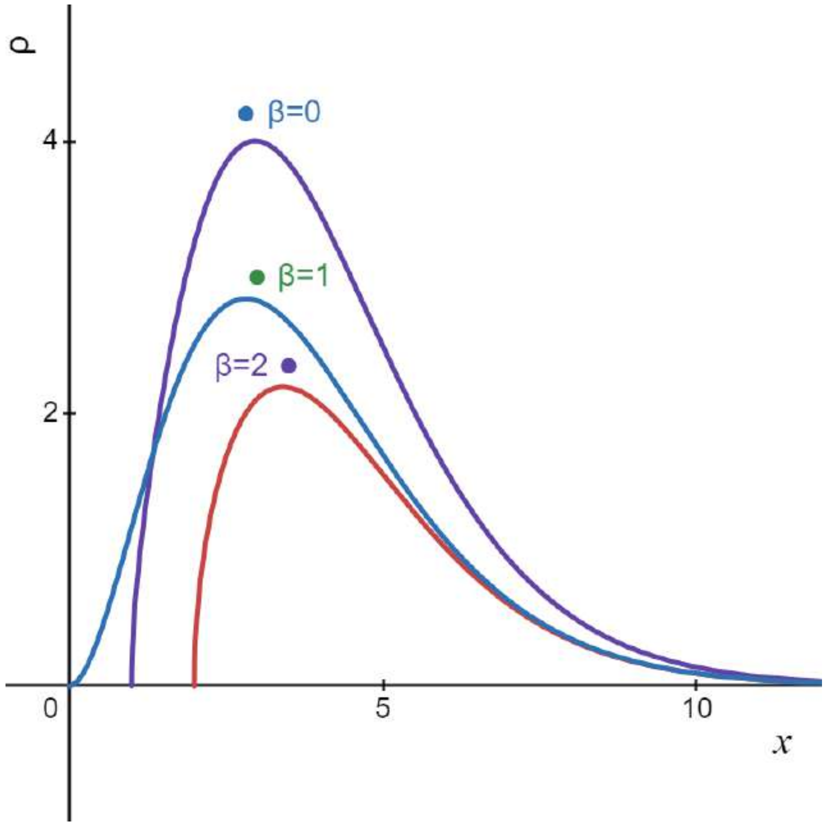

Figure 1. Modified Planck's laws (MPL)

The ratio of the average equivalent mass of a photon to its rest mass $\frac{\langle m \rangle}{m_\gamma}$ can be written simply as $\frac{\langle \epsilon \rangle}{\beta kT}$, from which it can be seen that for $\beta \ll 1$ (in the case of high temperatures) it tends to infinity (and the law of blackbody radiation tends to the standard Planck's law), which can formally be interpreted as a decrease in the photon's rest mass or an increase in the average mass equivalent of the photon while maintaining the rest mass. More detailed behavior of this function at small and large values of the parameter β is discussed in the Appendix. Correspondingly, it also determines the difference between the modified Stefan-Boltzmann law in these extreme cases and its standard analogue. At small values of the parameter $\beta \ll 1$, the average mass of the photon is defined by formula (60) of the Appendix, which gives:

$$\langle m \rangle = \frac{6.493939 - 0.822407\beta^2}{2.404112 - 0.5\beta^2(1 + \beta - ln\beta)} \frac{kT}{c^2}, \tag{15}$$

where the expression dependent on β is simply the ratio of integrals in (8), for which we use the notation $\alpha(\beta)$ in Appendix. For large values of β, the mass ratio is given by the last formula (67)

$$\frac{\langle m \rangle}{m_\gamma} = \frac{\alpha(\beta)}{\beta} \approx 1. \tag{16}$$

Hence it follows that the average photon mass equivalent is the upper bound for the photon's rest mass (which is to be expected), and their values approach each other at low temperatures. In Table 1 we present the results of numerical integration of the mass ratio $r =< m > /m_\gamma$ for different values of β.

Table 1. Mass ratio vs. parameter β

β	0.005	0.05	0.5	5	50
r	540.2	54.12	5.622	1.361	1.039

We see how steeply the mean mass equivalent approaches the rest mass of the photon with an increase in β. Even for intermediate temperatures ($m_\gamma c^2 \sim kT$) these two masses are close enough to each other

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within the same order of magnitude. As we said above (and it also can be seen from equation (10)), for the standard Planck spectrum:

$$\langle m \rangle = 2.701 \frac{kT}{c^2} = 4.141 \times 10^{-40} T kg. \tag{17}$$

It is interesting to evaluate the mean mass equivalent of the photon for some temperatures that are of physical interest. The best approximation to absolute zero, achieved experimentally, is estimated to be equal to $T = 3.8 \times 10^(- 11 \circ K$ (Depner & et al., 2021). Then for the hypothetic radiation with such a low effective temperature

$$\langle m \rangle = 1.574 \times 10^{-50} kg. \tag{18}$$

For the CMB photon with the temperature of $T = 2.725 \circ K$ (Fixsen, 2009, Peebles, 2020), we have

$$\langle m \rangle = 1.128 \times 10^{-39} kg. \tag{19}$$

This number is expectedly different (as it follows from the equations shown above) from estimates of m, obtained by investigation of CMB dipole anisotropy (De Bernardis et al., 1984) with a confidence level of $68\% : m_\gamma \sim (2.9 \pm 0.1) \times 10^{-54} kg$. In the meantime, based on radio-wave interferometriv measurements of the free-space velocity of electromagnetic waves on 72 GHz (almost twice less than the frequency of CMB radiation, peaked at $160.2 GHz$),with the accuracy $\sim 10^{-7}$ (Froome, 1958). One can estimate the upper limit of $m_\gamma < 4.3 \times 10^{-43} kg$ (see Tu et al., 2005). These results simply point to the low value of β ($\beta \ll 1$) and show that CMB radiation is blackbody radiation by its nature, not noticeably affected by the massive photon if the temperature is high. It follows from (12) that the mass equivalent of the photon tends to infinity with unlimited increase in temperature. This is a purely formal outcome that cannot be satisfied because of the reasons explained below in the text.

The mass equivalent of the photon, used in this paper and earlier in Saiyan (2023), is defined in free space and does not have to be confused with the effective mass of the photon defined in quantum electrodynamics instead of the term relativistic mass. It comes into play due to the interaction of photons with a medium (such as plasma triggered by a strong laser pulse in the gas) (Emelyanov, 2017, Mendonca, 2001, Yee, 1984) and is determined through the frequency of plasma oscillations. These two concepts are the same only if the rest mass of the photon equals zero. The only similarity is that the effective mass at high temperatures (if the energy of radiation is much greater than the rest energy of an electron) is proportional to the temperature Emelyanov (2017), as we have stated above for the mass equivalent and the rest mass in formulas (14) and (15) for high and low temperatures. As we can conclude from (15), our calculations show that the massiveness of the photon seems to be more significant factor at low temperatures ($kT \ll m_\gamma c^2$) and is negligible for high temperatures when the MPL approaches the standard Planck law (low and high temperatures are linked to the independently estimated rest mass of the photon). This conclusion agrees with the statement of Primak & Sher (1980) according to which the photon would acquire a non-zero mass mγ below some low temperature ". This idea is in tune with the physical model of Resca (2023) for dark energy as a Bose-Einstein condensate of cosmologically massive photons. The process which is taking place for bosons under some critical temperature, but impossible for massless photons in free space. The Bose-Einstein condensate was achieved in 2010 in a microscopic optical cavity for photons with effective mass (Klaers et al., 2010).

2.2. Modified Stefan-Boltzmann Law

The modified Stefan - Boltzmann law can be obtained by defining the upper integral in (10) after substituting expression (9) into it and integrating over all frequencies. This gives the total radiation energy density across the spectrum:

$$u(T) = \frac{8\pi}{(hc)^3} (kT)^4 \int_\beta^\infty \frac{x^3}{e^x - 1} \sqrt{1 - (\frac{\beta}{x})^2} dx. \tag{20}$$

Following the integration method for $\beta \ll 1$, presented in the Appendix, it is possible to write the modified Stefan-Boltzmann law, which, unlike the standard one, contains a temperature-dependent correction factor (see also Nyambuya, 2017, Pardy, 2018, Torres-Hernandez, 1985).

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$$u(T) = \sigma T^4[1 - \frac{5}{4(\pi)^2}(\frac{m_\gamma c^2}{kT})^2], \tag{21}$$

in which the Stefan-Boltzmann constant is given by the expression $\sigma = \frac{8}{15}\frac{(\pi)^5}{(hc)^3}k^4$. At high β values (that is, at low temperatures $kT \ll m_\gamma c^2$) the dependence of the total radiation density on temperature already changes. Returning to the first formula in (A.16) in Appendix we find:

$$u(T) = \frac{8\pi}{(hc)^3}\sqrt{\frac{\pi}{2}}(m_\gamma c^2)^4(\beta)^{\frac{5}{2}}e^{-\beta}. \tag{22}$$

That is, the radiation density at low temperatures is characterized by the presence of a rapidly decreasing exponential term and with a decrease in temperature tends to zero much faster than in the case of the standard Stefan-Boltzmann law. This is due to the fact that the condition $\beta \gg 1$ automatically entails $x \gg 1$, which means a transition to the Wien's radiation law. In the case of the Raleigh-Jeans law, the condition $x \ll 1$ automatically entails the condition $\beta ll1$, but the converse is no longer true: $x \ll 1$ does not follow $\beta \ll 1$. As we can see, a massive photon gas has a more complex temperature behavior than a massless one. This circumstance also affects the behavior of radiation pressure in different temperature regimes.

2.3. Modified Wien's Displacement Law

This aspect of the theory of massive photon radiation has not been discussed in the literature, as far as we know, but it is an important component of the theory of blackbody radiation. If we take the derivative of function (11) with respect to the dimensionless variable x, we arrive at the equation $(3x^2 - \beta^2)(e^x - 1) = x(e^x)(x^2 - \beta^2)$, which can be conveniently transformed to the form:

$$e^x[x^2(3 - x) + \beta^2(x - 2)] = 3x^2 - 2\beta^2. \tag{23}$$

This coincides with the standard equation for finding the maximum frequency of the standard Planck's law for $\$beta = 0$

$$e^x(3 - x) = 3, \tag{24}$$

that has two solutions: the trivial $x = 0$, which defines the minimum of radiation at $\nu = 0$ and $x_m = 2.821$, which defines the maximum. The frequency of radiation maximum for $\beta > 0$ can be found from the following relation:

$$\nu_m = x_m \times \frac{kT}{h} = 2.08366 \times 10^{10} T\frac{Hz}{K}x_m, \tag{25}$$

where values of x_m can be computed numerically for arbitrary β. Some examples are shown in Table 2.

Table 2. Parameters of MPL

0	2.82144	5.87892
1	2.96720	6.18264
2	3.40845	7.10205
3	4.09228	8.52692
4	4.91805	10.22670
5	5.81583	12.09740
6	6.75108	14.06700
10	10.63470	22.15760
15	15.58460	32.46840
20	20.56150	42.84320
25	25.54840	53.23420
30	30.53980	63.63460
50	50.52330	105.2730

Hence it follows that the maximum of the radiation of massive photons with decreasing temperature (increasing β) has a greater shift to the high-frequency part of the spectrum compared to the standard

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Planck's law. One can easily see that for large values of β (low temperatures) $x_m \approx \beta$ (more precisely $x_m \approx \beta + 0.5$), which gives us the equation

$$x_m = 4.8 \times 10^{-11} \frac{\nu_m}{T} \frac{K}{Hz} \approx \beta = \frac{m_\nu c^2}{kT}. \tag{26}$$

From the last relation, we have

$$m_\gamma \approx 4.8 \times 10^(-11) \frac{k}{c^2} \nu_m = 8.668 \times 10^{-51} \nu_m \frac{kg}{Hz} \tag{27}$$

This relation shows that the ratio $\frac{m_\gamma}{\nu_m}$ remains constant at low temperatures and equal to $8.668 \times 10^{-51} \frac{kg}{Hz}$. The lowest frequencies of electromagnetic radiation, discussed in publications, are around $10^{-3} Hz$ (estimated for the Earth's magnetic field) and known as micro-pulsations (Troitskaya & Gul'elmi, 1967). Bass & Schredinger (1955)consider the micro-pulsations as Gaussian fluctuations of the permanent magnetic field. If we substitute this number into the last relation instead of ν_m we obtain m_γ $8.668 \times 10^{-54} kg$, which is almost the same number as the one accepted by Physical Data Group (PDG) (Navas & and Particle Data Group, 2024) and considered to be the most reliable upper experimental limit for the rest mass of the photon.

2.4. Radiation pressure of amassive photon gas

The radiation pressure is determined by the formula p=u/3. In the explicit form

$$p = \frac{1}{3} \sigma T^4 [1 - \frac{5}{4\pi^2} \frac{m_\gamma c^2}{(kT)^2}], \tag{28}$$

if the high temperature condition is satisfied ($m_\gamma c^2 \ll kT$). At low temperatures

$$p = \frac{1}{3} \frac{8\pi}{(hc)^3} \sqrt{\frac{\pi}{2}} (m_\gamma c^2)^4 \beta)^{\frac{5}{2}} e^{-\beta}. \tag{29}$$

For high temperatures we have the following inequality:

$$T \gg 6.523 \times 10^{39} m_\gamma \, (K/kg). \tag{30}$$

The rest mass estimates, obtained from observations of different astrophysical sources in different wavelength ranges and under different conditions of the space environment vary within $(10^(-63) - 10^(-31))kg$ (Navas & and Particle Data Group, 2024, Shaefer, 1999, Tu et al., 2005). If we accept the upper limit for the rest mass of the photon suggested by PDG, then the temperature can be considered high if

$$T \gg 1.163 \times 10^{-14} \circ K. \tag{31}$$

This condition is met in the Universe with great excess. That is, virtually MPL does not differ noticeably from the standard Planck law as we stated above. But if the estimate of the photon rest mass is refined to a higher order of magnitude, then this conclusion must be revised. For example, if we accept $m_\gamma \approx 10^{-31} kg$ (based on studies of Mark 421 galaxy (see Shaefer, 1999, and references therein) as the upper limit of all estimates of the rest mass, then we must infer that the temperature is high if the following condition holds:

$$T \gg (10^8 - 10^9) \circ K \tag{32}$$

From the perspective of application of MPL it may affect some conclusions regarding stellar models. But is this upper limit of m_γ acceptable from a physical point of view?

3. Applications in astrophysics

3.1. Stellar models

As we know, the internal structure of stars is described by polytropic models, the stability of which is defined by the balance of gravitational forces and the net force of gas and radiant pressure, which depends on temperature. For stars such as the Sun, the Eddington model with a polytropic index of n=3 is considered

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the most acceptable (Eddington, 1926). In the model, the ratio of gas pressure to net gas and radiation pressure is assumed to be constant throughout the volume. It is obvious that due to properties of the massive photon gas set above, that criterion no longer holds. Estimates of the temperature in the nuclear region of the Sun lead to the values at which thermonuclear reactions begin, and the condition of high temperatures is excessively fulfilled even for values of m_γ of higher orders of magnitude. This allows to neglect the correction term in the expressions for the Stefan-Boltzmann law and radiant pressure. Therefore, in general, it can be assumed that conclusions, derived within the framework of conventional stellar models, remain valid and the massiveness of photons, by and large, does not play a significant role here. Note that the radiation pressure is much more significant in massive stars, where temperatures in nuclear regions can reach $T \sim 1.5 \times (10^8 - 10^9) \circ K$, and the main source of opacity of the medium is the Thomson scattering (Maeder et al., 2012). If the upper limit of m_γ, discussed above is accepted, then for temperatures just mentioned, $\beta \approx 1$, and correction factor to the Stefan-Boltzmann law and radiant pressure should be taken into consideration.

3.2. Radiation scattering

We will briefly discuss the effect of the "massiveness" of radiation on two types of scattering Thomson and Compton. Massive radiation is a solution of the Proca equation, in which the wave vector is represented in the form (5) or (6), reflecting the effect of optical dispersion in vacuum. We will limit ourselves to considering the classical case of the Thomson scattering at the qualitative level. Despite quantum theories are also available in publications (Crowley & Gregori, 2014), the massiveness of the photon is not considered there. Describing the scattering of a monochromatic electromagnetic wave on a non-relativistic electron, we consider the dependence of its phase velocity on the frequency, using it instead of the standard speed of light. As a rule, the dependence of the electric field strength on coordinates and the influence of the magnetic field strength vector on the expression for the Lorentz force are neglected due to the smallness of the electron's velocity compared to the speed of light. Taking into consideration the radiation reaction, the total cross section of Thomson scattering (in terms of the cyclic frequency of the electromagnetic wave ω takes the form Terletsky & Ribakov (1980):

$$\sigma_T(\omega) = \frac{\sigma_0}{1 + \frac{4(r_0)^2(\omega^2)}{9c^2}}, \tag{33}$$

where σ_0 is the cross-section of the classic Thomson scattering. In the absence of radiation reaction, it is independent of frequency ω: $\sigma_0 = \frac{8\pi}{3} r_0^2$. The phase velocity can be found from (5):

$$\sigma v_p h = \frac{\omega}{k} = c\frac{\omega}{\sqrt{\omega^2 - {\omega_0}^2}}. \tag{34}$$

Substituting this formula in the expression instead of the usual speed of light, we get:

$$\sigma_T(\omega) = \frac{\sigma_0}{1 + (\frac{4(r_0)^2}{9c^2})(\omega^2 - (\omega_0)^2)}. \tag{35}$$

It follows that the cross-section of the Thomson scattering of a massive electromagnetic wave on an electron is greater than the same cross-section in scattering of a massless wave. The cross-section increases with increase of the rest mass and energy of the photon. This increases the optical thickness of the medium, making it less permeable to massive radiation. In this case, the standard Thomson cross-section is reached not at $\omega = 0$, but at $\omega = \omega_0$. It is natural to expect that the same conclusion about the cross-section is true for Compton scattering as well, which transforms into Thomson scattering at low photon energy. It is well known that Compton scattering is incoherent, unlike Thomson scattering, and is characterized by a shift in the frequency of the scattered photon depending on the scattering angle towards lower values compared to its frequency before scattering due to the transfer of energy to the electron. This (Compton) shift is usually written in terms of wavelengths in the form:

$$\Delta\lambda = \lambda\prime - \lambda = \frac{h}{m_e c}(1 - \cos\theta), \tag{36}$$

where $\lambda_e, \lambda\prime$ are the wavelengths of the photon before and after scattering respectively, θ is the scattering angle, and $\lambda_e = \frac{h}{m_e c} h)$ is the electron's Compton wavelength. It can be shown that in the case of massive photons, this shift is slightly less than the one expected for zero-mass photons. This can be verified by

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writing down the laws of conservation of energy and linear momentum for this case of Compton scattering in the electron's rest system:

$$E_\gamma + E_e = E\prime_\gamma + E\prime_= E\prime_\gamma + \sqrt{(p\prime_e c)^2 + (m_e c)^2}), \tag{37}$$

$$p_\gamma + p_e = p'_\gamma + p'_e, \tag{38}$$

In the equations shown above E_γ, $\mathbf{p}_\gamma$ are the energy and linear momentum of the photon before scattering, E_e - electron rest energy. Energies and momenta of the photon and electron after scattering are provided with an apostrophe sign. Now we can find the difference in wavelengths of the incident and scattered photon (from the system of equations (36, 37) by substituting into it the expressions for specified quantities in the form:

$$\Delta\lambda = \lambda\prime - \lambda, \lambda = \frac{c}{\nu}, \lambda\prime = \frac{c}{\nu\prime}, \tag{39}$$

$$E_\gamma = h\nu, E_e = m_e c^2, E\prime'_\gamma = h\nu\prime, \tag{40}$$

$$p_e = 0, \quad p_\gamma = \hbar k, \quad p'_\gamma = \hbar k' \tag{41}$$

The magnitudes of wave vectors of incident k and scattered k'photon in (40) are defined by formula (5) for the optical dispersion in vacuum.

$$\Delta\lambda = \lambda\prime - \lambda = \lambda_e[1 - \sqrt{(1 - \frac{\lambda\prime^2}{\lambda_0^2})(1 - \frac{\lambda^2}{\lambda_0^2})}\cos\theta - \frac{\lambda\lambda\prime}{\lambda_0^2}]. \tag{42}$$

Here, $\lambda_0 = \frac{h}{m_\gamma c}$ is the photon ' s Compton wavelength. λ_0 is the limiting value of the photon wavelength, infinite if $m_\gamma = 0$. Obviously, in this case (41) follows the ordinary Compton formula. By substituting, $\lambda\prime = \lambda + \Delta\lambda$,(41) can be reduced to a form that determines the irreducible dependence of the Compton shift on the wavelength of the incoming photon, an effect that is missing from the ordinary Compton effect. The expression on the right side of 41) can be simplified considering that $\lambda\prime, \lambda \ll \lambda_0$. Neglecting terms of order of smallness higher than two, we obtain:

$$\Delta\lambda = \lambda_e 1 - [1 - \frac{1}{2\lambda_0^2}(\lambda\prime^2 + \lambda^2)\cos\theta - \frac{\lambda\lambda\prime}{\lambda_0{}^2}. \tag{43}$$

For $\theta = 0$ the equation (42) in terms of $\Delta\lambda$ takes the form of a quadratic equation

$$\Delta\lambda[1 - \frac{\lambda_e}{2\lambda_0^2}\Delta\lambda] = 0, \tag{44}$$

which has one physically acceptable solution $\Delta\lambda = 0$, the same as in the case of the ordinary Compton effect, which means that there is no change in the frequency of the photon as it is scattered forward. Further, in vertical scattering $\theta = \frac{\pi}{2}$ we have a linear equation the solution of which is

$$\Delta\lambda = \lambda_e \frac{1 - \frac{\lambda^2}{\lambda_0^2}}{1 + \frac{\lambda_e\lambda}{\lambda_0^2}}. \tag{45}$$

That is, the Compton shift is less than in the case of massless photons when it simply equals λ_e. At $\theta = \pi$ (backscattering) (42) is reduced to a quadratic equation with one physically acceptable solution:

$$\Delta\lambda \approx \lambda_e(1 - \frac{\lambda^2}{\lambda_0^2}) \le 2\lambda_e. \tag{46}$$

This leads to the same conclusion as for vertical scattering: the Compton shift for massive photons is smaller than for massless photons. The solution (45) slightly differs from backscattering case in the standard Compton-effect when $\Delta\lambda = 2\lambda_e$.

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3.3. Early stages of the Universe

We have already mentioned above that as the radiation temperature increases, the average mass equivalent of the Planck photon grows boundlessly. This result is formal for two reasons. The first one is that with an increase in temperature and, correspondingly, radiation density, the probability of two-photon and multiphoton collisions increases, leading to the production of electron-positron pairs at energies above 1.022 MeV, which limits the further growth of the photon's mass equivalent. The reaction of particle-antiparticle pair production, resulting from massless photons collision, is known as the Breit-Weeler process (Breit & Wheeler, 1934) (hereafter the BW-process) and is the opposite of the annihilation process. Experiments show very low probability of the phenomenon ($\sim (10^{-7} - 10^{-5})$ (Ribeyre & et al., 2016), depending on the energies of the photons), which has not been confirmed over almost eight more decades (Adam & et al., Adam & et al.). Collisions between photons lead to the "viscosity" of the photon gas and deviation from the condition of ideality. Speaking in general, the radiation of colliding photons cannot be described by Planck's standard law. The cross-section of BW-process is $\sim {r_0}^2$ (where $r_0 \approx 2.8179 \times 10^{-15} m$ is the classical electron radius) with an accuracy of up to a coefficient of the order of unity (Ribeyre & et al., 2016). At these distances, quantum electrodynamic effects become significant. The Compton wavelength of a massive photon reaches the value of r_0 at energies with the equivalent temperature $T \sim 10^1 2 \circ K$, which is typical for the transition from the quark epoch to the hadron epoch in the early universe.

The second reason is that it makes sense to talk about photons as independent particles and carriers of electromagnetic interaction up to temperatures not higher than $\sim 10^1 5 \circ K$. In this case, a spontaneous violation of the symmetry of electroweak interaction, associated with a drop in temperature as the Universe expands, takes place, after which the electromagnetic interaction separates from the weak one, and becomes a distinct physical force. This happens at the times of $\sim 10^{-12}$ seconds after the Big Bang and is interpreted as an electroweak phase transition due to the Higgs mechanism (Higgs 1964). In the case of photons with non-zero mass, as in Thomson scattering, it is natural to expect an increase in the scattering cross-section for the BW-process compared to massless photons, and, therefore, an increase in its probability. As a result, it will increase the production of electron-positron pairs. The last circumstance as it said above limits growth of the photon mass equivalent and contributes to an increase in the opacity of the cosmic medium for high-frequency radiation in the range of energy values that exceeds the excitation threshold of the BW process. It results in limitation of the spectral range of high energy photons in cosmological medium. But we must make the following remarks as well. With the exponentially growing volume of the Universe in the early inflation epoch (Guth, 1981, Linde, 1984, Tsujikawa, 2003) and the rapid increase in the rate of cooling because of the expansion, the annihilation process won't be able to compensate for the loss of photons due to BW-process which cannot be long lasting. Collisions between massive photons seem to have more chances to produce particle - antiparticle pairs compared with massless photons. This production will increase the role of Thomson scattering in the formation of opacity of the cosmological medium. In the meantime, the existence of more electrons caused by more frequent collisions of the photons increases the probability of the formation of neutral atoms and decreases the opacity. It is difficult to say to what extent this process in the presence of a massive photon field affects the estimate of duration of the pre-recombination period in the evolution of the Universe. We do not know any estimates regarding this matter. However, from the above said, one can assume that the onset of the post-recombination epoch could have occurred somewhat earlier than is generally believed.

It is interesting to notice that because always $m_\gamma \leq \langle m > \rangle$ (regardless the value of parameter β and the upper limit of the mass equivalent of the photon is defined by (16), for $T = 10^{15} \circ K$ we have $m_\gamma \leq 4.141 \times 10^{-25} kg$ - the mass nearly equal to the mass of the Higgs boson ($125.35 GeV = 2.231 \times 10^{-25} kg$, (ATLAS Collaboration, 2023) (125.11 GeV). The conclusion is that photons cannot have the rest mass greater than this. None of the existing measurements of this quantity are close to this limit.

3.4. Active galactic nuclei

The BW-process, described in the previous section, can also be implemented using the inverse Compton effect in the near-nuclear regions of active galaxies, such as Markarian 421 and 501, which demonstrate a high degree of activity across the spectrum, from radio to gamma rays (Acciari et al., 2020, Aharonyaan & et al., 1997, Carnerero et al., 2017). The activation of the BW - process in these sources is possible because of synchro-Compton scattering, when photons of synchrotron radiation sharply increase their energy during inverse Compton scattering on the relativistic electrons that generated them proportionally toγ^2, where

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γ 1000 is the Lorentz factor (Beckmann, 2006, Sunyaev, 1986). For example, radio frequencies can "migrate" to the ultraviolet region, and optical photons can "migrate" to the gamma range. The role of massive photons in these processes can be manifested not only in the existence of time dispersion and variations in the speed of light. These high energy photons will contribute to EBL (Extragalactic Background light, which is another component of the total diffuse cosmic radiation in addition to CMB), and that way, to the pair-production cutoff (Franceschini et al., 1994, Stecker & de Jager, 1997)which is comprehensively explained and discussed by (Franceschini, 2021) who also has mentioned that the number of such energetic photons is decreasing very sharply with energy $\propto \gamma^{-2,-3}$ at least).

4. Spreading time of the photon wave packet

The wave packet of a massive photon can be formally represented as a very concentrated superposition of quasi-monochromatic waves with very close frequencies. Since (as can be seen from (5)) the dependence of frequency on the wave vector is nonlinear, purely theoretically, there must be some spreading of the wave packet of a massive photon in free space. For the massless photon the wave packet in vacuum is infinite, that is, the photon does not spread out in free space. As is known (Zemtsov & Bychkov, 2005), the spreading time of the wave packet is

$$\tau = \frac{(\Delta x)^2}{d^2\omega/dk^2} \tag{47}$$

where Δx is its width. If $\Delta x \sim \lambda = \frac{c}{\nu} = \frac{2\pi c}{\omega}$, one can estimate τ by finding ω from the equation (4):

$$\omega = \sqrt{(kc)^2 + \omega_0^2}\omega_0 = 2\pi\nu_0. \tag{48}$$

Performing differentiation, we obtain

$$\frac{d^2\omega}{dk^2} = \frac{2\pi\nu^3}{\nu_0^2}, \quad \tau \sim \frac{2\pi\nu}{\nu_0^2} = \frac{2\pi\nu h^2}{m_\gamma{}^2 c^4} \tag{49}$$

Taking as above $m_\gamma < 10^{-18} eV = 1.783 \times 10^{-54} kg$, we obtain

$$\tau > 1.074 \times 10^8 \nu(s) \tag{50}$$

If the left-hand side of the inequality is taken to be equal to the age of the Universe in seconds,

$$t_u \sim 4.352 \times 10^{17}\, s, \tag{51}$$

then all photons with the frequency $\nu \geq 4.052 \times 10^9 Hz (\lambda \leq 7.4 cm)$ have wave packets with a spreading time greater than the age of the Universe. For CMB radiation $\nu = 1.602 \times 10^{11}\, Hz$ we have $\tau \sim 1.721 \times 10^{19}\, s$ which exceeds this age by almost 40 times. Virtually, in a very wide range of frequencies, which are of interest to astrophysics, the wave packet of a massive photon does not spread.

5. Conclusions and discussion

The paper derives Planck's law of radiation for massive photons. The dependence of the ratio of the mean mass equivalent of the photon to its rest (invariant) mass on different temperature regimes is studied. It tends to infinity at high temperatures and approaches unity at low temperatures. In the first case MPL approaches the standard Planck's law, and the massiveness of the photon becomes a nonsignificant factor in the radiation process. At intermediate and low temperatures, deviation of MPL from Planck's law increases, and the massiveness of the photon should be considered with the lowering intensity of the radiation. Stefan-Boltzmann and Wien's displacement laws for massive photons are derived, as well as an expression for radiation pressure, which takes different forms at high and low temperatures. The estimate of the Thomson cross-section of the scattering of massive photons on a free electron turned out to be slightly greater than its value in the usual case, which increases the optical thickness of the scattering medium. In the case of Compton scattering, the wavelength shift is less than its value in the process of massless photons, except in the case of scattering forward, where the wavelength remains unaltered. Apparently, the existence of the rest mass of the photon ($m_\gamma < 10^{-54} kg$) is not a significant factor for the wide range of astrophysical events.

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However, this conclusion may change with mass estimates of a higher order of magnitude. Estimates of the spreading time of the massive photon show that they remain stable in free space at time intervals exceeding the age of the Universe. Slight deviations of the Thomson and Compton scattering cross-sections (as well as Compton shift) from their standard values could serve as indirect evidence in favor of the hypothesis that the photon has a rest mass. As we know, these deviations have not been discovered so far.

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Appendices

Appendix A Analytical study of integrals

Integrals in (10) can analytically be evaluated in two extreme cases, low and high temperatures (i.e., at very large and small β). Torres-Hernandez (1985) discussed these extreme physical cases just in a few words not presenting any detailed calculations. Although it is obvious that at $\beta \ll 1$ MPL approaches the standard Planck's law, it is nevertheless interesting to clarify the expression for the mean mass equivalent of the photon at non-zero values of this parameter. It makes theoretical analysis of the problems considered here more complete. Omitting the constant factors in the upper and lower integrals in (10) and introducing notations $I_1(\beta), I_2(\beta)$ for them, we write it down in terms of the variable x and parameter β, introduced earlier in the text:

$$I_1(\beta) = \int_{\beta}^{\infty} \frac{x^3}{e^x - 1}\sqrt{1 - (\frac{\beta}{x})^2} dx \tag{52}$$

$$I_(2)(\beta) = \int_{\beta}^{\infty} \frac{x^2}{e^x - 1}\sqrt{1 - \frac{\beta}{x}^2} dx \tag{53}$$

Decomposing the first integral into the Taylor series by powers of β and applying Leibniz's formula, we get, limiting ourselves to the first two terms,

$$I_(1)(\beta) \approx \zeta(4)\Gamma(4) - 1/2\zeta(2)\beta^2, \tag{54}$$

where $\zeta(s)$ is the Riemann zeta function, and $\Gamma(x)$ is the gamma function (Batman & Erdelyi, 1974):

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx \tag{55}$$

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt. \tag{56}$$

To estimate the second integral in (52), it is convenient to decompose the square root in the Tylor series by powers of *beta* in the vicinity of the point $\beta = 0$, which leads to the difference between the two integrals:

$$I_2(\beta) \approx \int_{\beta}^{\infty} \frac{x^2}{e^x - 1} dx - \frac{1}{2}\beta^2 \int_{\beta}^{\infty} \frac{dx}{e^x - 1}, \tag{57}$$

if we confine ourselves, as before, to the first two terms. The first integral in (57) can be written as

$$\int_{\beta}^{\infty} \frac{x^2}{e^x - 1} dx = \int_0^{\infty} \frac{x^2}{e^x - 1} dx - \int_0^{\beta} \frac{x^2}{e^x - 1} dx \approx \zeta(3)\Gamma(3) - \frac{1}{2}\beta^2, \tag{58}$$

using the fact that with small β in the last integral in (58) $e^x - 1 \approx x$. In the last integral in (57) one can make a substitution $t = e^x - 1$, and making β -$ln(\beta)$. Thus

$$I_2(\beta) \approx \zeta(3)\Gamma(3) - \frac{1}{2}\beta^2(1 + \beta - ln\beta). \tag{59}$$

Hence it follows that at high temperatures

$$\langle m \rangle = \frac{\zeta(4)\Gamma(4) - \frac{1}{2}\zeta(2)\beta^2}{\zeta(3)\Gamma(3) - \frac{1}{2}\beta^2(1 + \beta - ln(\beta))} \frac{kT}{c^2}. \tag{60}$$

This expression leads to (14) after substituting numerical values of the zeta and gamma functions. The expression for $I_1(\beta)$ leads to (20) and (27), because

$$I_(1)(\beta) = \frac{\pi^4}{15}(1 - \frac{5\beta^2}{4\pi^2}). \tag{61}$$

To analyze integrals $I_1(\beta)$, $I_2(\beta)$ at low temperatures, note that if $\beta \gg 1$, then $x \gg 1$ as well. Therefore, $e^x \gg 1$, and $e^x - 1 \approx e^x$. Then, if we set $t = \frac{x}{\beta}$, the integral $I_1(\beta)$ can be rewritten in the form:

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$$I_1(\beta) = \beta^4 \int_1^\infty t^2 e^{-\beta t} \sqrt{t^2-1} dt, \tag{62}$$

which can be expressed in terms of the Macdonald function $K_\nu(\beta)$ using the following representation Batman & Erdelyi (1974):

$$\Gamma(\nu+\frac{1}{2})K_\nu(z) = \sqrt{\pi}\frac{z}{2}^\nu \int_1^\infty e^{-\beta t}(t^2-1)^{\nu-\frac{1}{2}} dt, \nu > -\frac{1}{2}, Rez > 0, \tag{63}$$

where ν is the order of the Macdonald function. In our case $\nu = 1, Rez = \beta \gg 1$. It gives

$$\frac{K_1(\beta)}{\beta} = \int_1^\infty e^{-\beta t}\sqrt{t^2-1} dt. \tag{64}$$

Since $t^2 = (t^2-1)+1$, for (62) we have:

$$I_1(\beta) = \beta^2[K_2(\beta) + \beta K_1(\beta)]. \tag{65}$$

With the aim of (64) the integral (62) can be expressed in the following way:

$$I_2(\beta) = \beta^3 \int_1^\infty t e^{-\beta t}\sqrt{t^2-1} dt = -\beta^3 \frac{K_1(\beta)}{\beta}\beta. \tag{66}$$

For $\beta \gg 1$ $K_n u(\beta) \sim \sqrt{\frac{\pi}{2\beta}} e^{-\beta}$ and then

$$I_1(\beta) \approx \sqrt{\pi} 2\beta^{\frac{5}{2}} e^(-\beta), \quad I_2(\beta) \approx \sqrt{\pi} 2\beta^{\frac{3}{2}} e^{-\beta}, \quad \alpha(\beta) = \frac{I_1(\beta)}{I_2(\beta)} \approx \beta. \tag{67}$$

Hence it follows the formula (15) in the text.

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