The eigenfunctions method in relation to diffuse reflection problems of the radiative energy transfer theory

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Abstract

Using the new method proposed earlier by the author, a solution of diffuse reflection problem is given in this report. A case of diffuse reflection of radiation from a semi-infinite absorbing and isotropically reflecting medium is presented, when in the elementary act of scattering there is a general law of redistribution of radiation by frequencies. In the approach presented here, conventionally called the method of "decomposition of the resulting field" - DRF, in contrast to the previously widely used traditional method of "decomposition of a single act" of scattering - DSA, the problem of diffuse reflection is solved without the use of any decomposition or special representation of the characteristic of the elementary act of scattering, i.e., the redistribution function of radiation by frequencies. The resulting radiation field, formed in the medium by means of multiple scatterings, is sought and constructed directly in the form of a bilinear series through a specially derived system of its own eigenfunctions. The physical basis of this approach lies in the fact that after each successive act of scattering, the field formed in the medium becomes more and more smoothed, so its direct representation in a decomposed form is more expedient than the traditionally used decomposition of a pre-known "unsmoothed" characteristic of the elementary act of scattering.

Keywords: *radiative transfer, diffuse reflection problem, partial redistribution of radiation by frequencies, separation of variables, nonlinear Ambartsumian's functional equation, eigenfunctions and eigenvalues problem*

1. Introduction and purpose of the report.

The problem of diffuse reflection is one of the classical problems of theoretical astrophysics, which also has a wide application value in various related fields of science. Its solution in the case of a semi-infinite scattering-absorbing medium under isotropic monochromatic scattering by introducing the principle of invariance (Ambartsumian (1942, 1943a,b, 1944a,b,c)) was obtained in the form of

$$\rho\left(\mu,\mu'\right)=\frac{\lambda}{2\mu'}\frac{\varphi\left(\mu\right)\varphi\left(\mu'\right)}{\frac{1}{\mu}+\frac{1}{\mu'}},\varphi\left(\mu\right)=1+\int_0^1\rho\left(\mu,\mu'\right)d\mu',\tag{1}$$

$$\varphi\left(\mu\right)=1+\frac{\lambda}{2}\mu\varphi\left(\mu\right)\int_0^1\frac{\varphi\left(\mu'\right)}{\mu+\mu'}d\mu'.\tag{2}$$

Here's $\rho\left(\mu,\mu'\right)d\mu$ the probability of diffuse reflection of the quantum from the medium in the direction of μ (cosine of the angle of the quantum leaving the medium in relation to the normal to its boundary), in the solid angle of $2\pi d\mu$ on condition that it enters the medium from the direction μ' . The value of λ indicates the probability of survival of the quantum in the elementary act of scattering or the so-called albedo of a single scattering. In solution (1) there is a phenomenon of separation of angular variables, i.e., the desired function of many (here – two) independent variables is expressed through a function of a smaller number (here - one) of the angular variable. Later, Sobolev showed that a separation of the same type takes place for frequency variables in the case of a complete redistribution of radiation over frequencies in isotropic scattering of quanta (see, e.g., (Sobolev, 1963)). In the case of anisotropic monochromatic scattering, the separation of angular variables was first achieved in the work (Ambartsumian, 1943b, 1944b). For this purpose, the scattering indicatrix was preliminarily decomposed according to Legendre polynomials (Ambartsumian, 1941). Then, the Fourier cosine expansion was performed along the azimuths, and for the azimuthal harmonics of the scattering indicatrix, a bilinear expansion was obtained by the associated Legendre functions

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$$\chi^m(\mu,\mu') = \sum_{i=m}^{N} c_i^m P_i^m(\mu) P_i^m(\mu'). \quad (3)$$

Substituting this decomposed form of the scattering indicatrix into the Ambartsumian's functional equation, after simple calculations, leads to the solution of the diffuse reflection problem in an explicit form. The azimuthal harmonic of the diffuse reflection function, which depends on two independent angular variables, is explicitly expressed through the system of auxiliary functions of one angular variable. These auxiliary functions were derived from a system of functional nonlinear integral equations. The number of equations in this system was naturally equal to the number of terms in the bilinear expansion of azimuthal harmonics of the scattering indicatrix (3) by the associated Legendre functions. Later, when considering the problem of diffuse reflection in the case of incoherent scattering, under the general law of redistribution of radiation over frequencies - $r(x,x')$, in order to achieve the separation of frequency variables (x,x'), the special representation of the characteristics of elementary act of scattering was also used. The function of redistribution of radiation by frequencies was given by a bilinear series of a specially constructed system of its eigenfunctions ((Engibaryan, 1971, Engibaryan & Nikogosyan, 1972))

$$r(x,x') = \sum_{i=1}^{N} A_i \alpha_i(x) \alpha_i(x') \quad (4)$$

In this way, it was possible to achieve the separation of frequency variables. In some real-world applications, however, a much larger number of terms (on the order of 100 or more) must be taken into account in series (3) or (4), in order to achieve the necessary accuracy (see, e.g., Smokity & Anikonov (2008)). In this case, the calculation of the diffusely reflected field by the above-mentioned "traditional" method of decomposition of the characteristics of the elementary act of scattering will obviously be very difficult or inexpedient due to the corresponding increase in the number of equations in the system, which is subject to decision. The purpose of this report is to briefly present the result of solving the problem of diffuse reflection obtained using a new original method recently introduced by the author in his works (Pikichyan (2023a,b,c)). The expediency of the above method is illustrated below by analyzing the problem of diffuse reflection from a semi-infinite medium under isotropic incoherent scattering in the case of the general law of radiation redistribution over frequencies. It is shown that, in contrast to the hitherto known methods of analysis, the problem of diffuse reflection can be solved without any decomposition or special representation of the characteristics of the elementary act of scattering. Here, an explicit solution to the problem is sought directly in a decomposed form, through a specially constructed system of eigenfunctions that depend on a smaller number of independent variables. In order to find these eigenfunctions and their eigenvalues together, a general system of nonlinear integral functional equations and nonlinear algebraic equations is obtained. Such an approach, in contrast to the well-known methods of "decomposition of a single act" (DSA), should be conventionally called the method of "decomposition of the resulting field" (DRF).

2. Physical basis and expediency of finding a solution to the problem by means of DRF

It is obvious that in the process of multiple scattering of light in a medium, after each successive act of scattering, the radiation field in it becomes more and more smooth. The reason is that the function describing the previous distribution of the radiation field after each new scattering act is once again integrated with the nucleus of the single scattering act. It follows that the representation of the characteristics of this "smoothed" resulting field formed in process of multiple scatterings by a bilinear series will be more efficient. I.e., in the end, the same accuracy will be provided by a smaller number of series terms than in the case of decomposition of the original "unsmoothed" field of a single scattering (Pikichyan (2023a,b,c)).

3. Mathematical formulation of the problem

To solve the problem of diffuse reflection in the case of isotropic scattering, we previously obtained an explicit expression (Pikichyan (1978, 1980)):

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$$\rho\left(x,\mu;x',\mu'\right)=\frac{\lambda}{2\mu'}\frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\varphi\left(x'';x,\mu\right)r\left(x'',x'''\right)\varphi\left(x''';x',\mu'\right)dx''dx'''}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}, \tag{5}$$

$$\varphi\left(x',x;\mu\right)=\delta\left(x'-x\right)+\int_0^1\rho\left(x,\mu;x',\mu'\right)d\mu', \tag{6}$$

$$\varphi\left(x',x;\mu\right)=\delta\left(x'-x\right)+\frac{\lambda}{2\mu'}\int_{-\infty}^{+\infty}\varphi\left(x'';x,\mu\right)dx''\int_{-\infty}^{+\infty}r\left(x'',x'''\right)dx'''\int_0^1\frac{\varphi\left(x''';x',\mu'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}d\mu'. \tag{7}$$

Where is the desired diffuse reflection function $\rho\left(x,\mu;x',\mu'\right)$ dependent on the four independent variables is explicitly expressed in terms of an auxiliary function $\varphi\left(x',x;\mu\right)$ of a smaller number i.e., three, of independent variables. It is obvious that the decrease in the number of angular variables, like in the case (1), is also due to the isotropy of the elementary act of scattering. As for the frequency variables (the value $\alpha\left(x\right)$ represents the absorption profile), the property of separation does not take place for them, due to the absence of it in the elementary act of scattering $r\left(x'',x'''\right)$. It has been already pointed out above that in order to achieve the separation of frequency variables, the DSA method has been traditionally used, i.e., the method of decomposition of the characteristic of an elementary act of scattering in the form (4). Indeed, by substituting the expansion (4) into the relation (5), the well-known solution is directly obtained:

$$\rho\left(x,\mu;x',\mu'\right)=\frac{\lambda}{2\mu'}\frac{\sum_{i=1}^{N}A_i\varphi_i\left(x,\mu\right)\varphi_i\left(x',\mu'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}, \tag{8}$$

$$\varphi_i\left(x,\mu\right)\equiv\int_{-\infty}^{+\infty}\alpha_i\left(x'\right)\varphi\left(x';x,\mu\right)dx', \tag{9}$$

$$\varphi_i\left(x,\mu\right)=\alpha_i\left(x\right)+\frac{\lambda}{2}\sum_{i=1}^{n}A_i\varphi_i\left(x,\mu\right)\int_{-\infty}^{+\infty}\alpha_i\left(x'\right)dx'\int_0^1\frac{\varphi_i\left(x',\mu'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}\frac{d\mu'}{\mu'}. \tag{10}$$

Our task is to simplify the solution (5) by the DRF method, achieving the separation of frequency variables without involving any expansions or special representations of the elementary act of scattering.

It is well known that from the symmetry of the elementary act of scattering follows the symmetry of the diffuse reflection function itself

$$r\left(x'',x'''\right)=r\left(x''',x''\right)\Rightarrow\rho\left(x,\mu;x',\mu'\right)\mu'=\rho\left(x',\mu';x,\mu\right)\mu, \tag{11}$$

then the function introduced by notation

$$K\left(x,\mu;x',\mu'\right)\equiv\mu'\left[\frac{\alpha\left(x\right)}{\mu}+\frac{\alpha\left(x'\right)}{\mu'}\right]\rho\left(x,\mu;x',\mu'\right) \tag{12}$$

will also be symmetrical

$$K\left(x,\mu;x',\mu'\right)=K\left(x',\mu';x,\mu\right). \tag{13}$$

For this positive and symmetric kernel, we can formulate a problem for eigenvalues ν_j and eigenfunctions $\beta_j\left(x,\mu\right)$:

$$\nu_j\beta_j\left(x,\mu\right)=\int_{-\infty}^{+\infty}\int_0^1K\left(x,\mu;x',\mu'\right)\beta_j\left(x',\mu'\right)d\mu'dx', \tag{14}$$

with orthonormalization condition

$$\int_{-\infty}^{+\infty}\int_0^1\beta_i\left(x,\mu\right)\beta_j\left(x,\mu\right)d\mu dx=\delta_{ij}. \tag{15}$$

If, in some approximation N , the nucleus $K\left(x,\mu;x',\mu'\right)$ to be replaced by a bilinear series of its own eigenfunctions

$$K\left(x',\mu';x,\mu\right)\backsim K_N\left(x,\mu;x',\mu'\right)=\sum_{j=1}^{N}\nu_j\beta_j\left(x,\mu\right)\beta_j\left(x',\mu'\right), \tag{16}$$

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then, taking into account (12) and (16), an explicit solution to the initial problem of diffuse reflection in the same approximation N is obtained

$$\rho\left(x,\mu;x',\mu'\right)=\frac{1}{\mu'}\frac{\sum_{i=1}^{N}\nu_i\beta_i\left(x,\mu\right)\beta_i\left(x',\mu'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}. \tag{17}$$

Here we have the desired separation of variables (x',μ') of quanta entering the medium from variables (x,μ) of quanta leaving the medium. Thus, the problem was reduced to finding the eigenfunctions $\beta_i\left(x,\mu\right)$ and the eigenvalues of ν_i of the unknown kernel $K\left(x,\mu;x',\mu'\right)$.

4. Construction of an analytical solution to the problem using the DRF method

From the ratios (5), (6) and (12) follow the expressions

$$\frac{2}{\lambda}K\left(x,\mu;x',\mu'\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\varphi\left(x'';x,\mu\right)r\left(x'',x'''\right)\varphi\left(x''';x',\mu'\right)dx''dx''', \tag{18}$$

$$\varphi\left(x';x,\mu\right)=\delta\left(x'-x\right)+\int_0^1\frac{K\left(x,\mu;x',\mu'\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x')}{\mu'}}\frac{d\mu'}{\mu'}, \tag{19}$$

with the help of which it is not difficult to obtain a functional equation for the nucleus $K\left(x,\mu;x',\mu'\right)$

$$\frac{2}{\lambda}K\left(x,\mu;x',\mu'\right)=r\left(x,x'\right)+$$

$$\int_{-\infty}^{+\infty}r\left(x,x''\right)dx''\int_0^1\frac{K\left(x',\mu';x'',\mu''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}+\int_{-\infty}^{+\infty}r\left(x'',x'\right)dx''\int_0^1\frac{K\left(x,\mu;x'',\mu''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}+$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_0^1\frac{K\left(x,\mu;x'',\mu''\right)}{\frac{\alpha(x)}{\mu}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}r\left(x'',x'''\right)\int_0^1\frac{K\left(x',\mu';x''',\mu'''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x''')}{\mu'''}}\frac{d\mu'''}{\mu'''}dx''dx'''. \tag{20}$$

Thus, the eigenvalue and eigenfunctions problem (14) – (15) must be solved for the unknown function $K\left(x,\mu;x',\mu'\right)$, which satisfies to the functional equation (20).

Using the integral operator $\int_{-\infty}^{+\infty}\int_0^1\ldots\beta_j\left(x',\mu'\right)d\mu'dx'$ to the equation (20), a system of nonlinear algebraic an algebraic system of nonlinear integral equations is obtained

$$\frac{2}{\lambda}\nu_j\beta_j\left(x,\mu\right)=Z_j\left(x\right)+\sum_{k=1}^{N}\nu_kD_{kj}\left(x,\mu\right)+\sum_{k=1}^{N}\sum_{l=1}^{N}\nu_k\nu_lV_{klj}\left(x,\mu\right). \tag{21}$$

The following notations are accepted here:

$$Z_j\left(x\right)\equiv\int_{-\infty}^{+\infty}r\left(x,x'\right)\overline{\beta}_j\left(x'\right)dx',\quad\overline{\beta}_j\left(x'\right)\equiv\int_0^1\beta_j\left(x',\mu'\right)d\mu',$$

$$D_{kj}\left(x,\mu\right)\equiv\int_{-\infty}^{+\infty}r\left(x,x''\right)w_{kj}\left(x''\right)dx''+\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}w_k\left(x'',x,\mu\right)r\left(x'',x'\right)\overline{\beta}_j\left(x'\right)dx'dx'',$$

$$V_{klj}\left(x,\mu\right)\equiv\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}w_k\left(x'',x,\mu\right)r\left(x'',x'''\right)w_{lj}\left(x'''\right)dx''dx''',$$

$$w_{kj}\left(x''\right)\equiv\int_{-\infty}^{+\infty}\int_0^1w_k\left(x'',x',\mu'\right)\beta_j\left(x',\mu'\right)d\mu'dx'\ ,\quad w_k\left(x'',x',\mu'\right)=\beta_k\left(x',\mu'\right)\int_0^1\frac{\beta_k\left(x'',\mu''\right)}{\frac{\alpha(x')}{\mu'}+\frac{\alpha(x'')}{\mu''}}\frac{d\mu''}{\mu''}. \tag{22}$$

The use of the operator $\int_{-\infty}^{+\infty}\int_0^1\ldots\beta_j\left(x',\mu'\right)d\mu'dx'$ by the relation (21) taking into account the normalization condition (15), a system of nonlinear algebraic equations is obtained

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$$\frac{2}{\lambda}\nu_i = b_i + \sum_{k=1}^{N}\nu_k c_{ki} + \sum_{k=1}^{N}\sum_{l=1}^{N}\nu_k\nu_l f_{kli}, \tag{23}$$

where free terms in equations and coefficients for eigenvalues are given by notation:

$$b_i = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\overline{\beta}_i(x)\, r(x,x')\,\overline{\beta}_i(x')\,dx'dx,$$

$$c_{ki} = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_{ki}(x'')\, r(x'',x')\,\overline{\beta}_i(x')\,dx'dx'',$$

$$f_{kli} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w_{ki}(x'')\, r(x'',x''')\, w_{li}(x''')\,dx''dx'''. \tag{24}$$

Thus, using the DRF method, the problem of determining the desired eigenvalues ν_j and eigenfunctions $\beta_j(x,\mu)$ of the core $K(x,\mu;x',\mu')$ is reduced to the solution of a general system consisting of two more partial subsystems: nonlinear integral equations (21) and algebraic nonlinear equations (23).

5. General scheme of numerical calculation

To implement the numerical calculation as the initial zero approximation $\beta_i^{(0)}(x,\mu)$, we can take some system of orthonormalized polynomials, for example, Hermite, and use notation (24) to compute zero approximations of the remaining quantities:

$$\beta_i^{(0)}(x,\mu) \;\to\; b_i^{(0)},\quad c_{ki}^{(0)},\quad f_{kli}^{(m)},\quad \nu_i^{(0)} \equiv b_i^{(0)}. \tag{25}$$

Then, using (25), the first approximation of the eigenvalues of $\nu_i^{(1)}$ is calculated by applying the formula

$$\nu_i^{(1)} = \frac{\lambda}{2}\left[b_i^{(0)} + \sum_{k=1}^{N}\nu_k^{(0)}c_{ki}^{(0)} + \sum_{k=1}^{N}\sum_{l=1}^{N}\nu_k^{(0)}\nu_l^{(0)}f_{kli}^{(0)}\right] \tag{26}$$

After that, by values (22) and $\nu_i^{(1)}$:

$$\nu_i^{(1)},\quad \beta_i^{(0)}(x,\mu) \;\to\; Z_i^{(0)}(x)\,,\quad D_{ki}^{(0)}(x,\mu)\,,\quad V_{kli}^{(0)}(x,\mu) \tag{27}$$

the first approximation of eigenfunctions is calculated using the formula

$$\beta_i^{(1)}(x,\mu) = \frac{\lambda}{2\nu_i^{(1)}}\left[Z_i^{(0)}(x) + \sum_{k=1}^{N}\nu_k^{(1)}D_{ki}^{(0)}(x,\mu) + \sum_{k=1}^{N}\sum_{l=1}^{N}\nu_k^{(1)}\nu_l^{(1)}V_{kli}^{(0)}(x,\mu)\right], \tag{28}$$

and then the scheme is consistently applied

$$\nu_i^{(m+1)} \to \beta_i^{(m+1)}(x,\mu) \to \nu_i^{(m+2)} \to \beta_i^{(m+2)}(x,\mu) \to \cdots \to \nu_i^{(M)} \to \beta_i^{(M)}(x,\mu)\,. \tag{29}$$

Successive iterations continue at $m = 1,\ 2,\ 3,\ldots$ and at each successive value of m, the accuracy of convergence is evaluated in a standard way until the M - th iteration provides the necessary accuracy. Thus, the joint iterative scheme described above has the form of

$$\begin{gathered}\beta_i^{(m+1)}(x,\mu) = \tfrac{\lambda}{2\nu_i^{(m+1)}}\left[Z_i^{(m)}(x) + \textstyle\sum_{k=1}^{N}\nu_k^{(m+1)}D_{ki}^{(m)}(x,\mu) + \sum_{k=1}^{N}\sum_{l=1}^{N}\nu_k^{(m+1)}\nu_l^{(m+1)}V_{kli}^{(m)}(x,\mu)\right],\\ \nu_i^{(m+1)} = \tfrac{\lambda}{2}\left[b_i^{(m)} + \textstyle\sum_{k=1}^{N}\nu_k^{(m)}c_{ki}^{(m)} + \sum_{k=1}^{N}\sum_{l=1}^{N}\nu_k^{(m)}\nu_l^{(m)}f_{kli}^{(m)}\right].\end{gathered} \tag{30}$$

6. Relationship between the auxiliary functions of the DSA method and the eigenfunctions of the DRF method

By comparing the explicit analytical solutions of the same diffuse reflection problem obtained by the DSA-(8) and DRF-(17) methods, it is not difficult to write a two-way relationship between the values sought with their help: the auxiliary functions $\varphi_i(x,\mu)$ of the DSA method and the eigenfunctions - $\beta_j(x,\mu)$ with the eigenvalues ν_j of the DRF method (see Pikichyan (2023b)).

a) Suppose the values of $\varphi_i(x,\mu)$ are known, but it is necessary to find the values $\beta_j(x,\mu)$ and ν_j with their help:

$$\beta_j(x,\mu) = \frac{\sum_{i=1}^{N} A_i \varphi_i(x,\mu) q_{ij}}{\sum_{i=1}^{N} A_i q_{ij}^2}, \quad q_{kj} = \frac{\sum_{i=1}^{N} A_i a_{ik} q_{ij}}{\sum_{i=1}^{N} A_i q_{ij}^2}, \quad \nu_j = \frac{\lambda}{2}\sum_{i=1}^{N} A_i q_{ij}^2. \tag{31}$$

b) Assuming the values $\beta_j(x,\mu)$ and ν_j, it is necessary to use them to find the $\varphi_i(x,\mu)$

$$\varphi_i(x,\mu) = \alpha_i(x) + \sum_{j=1}^{N} \nu_j \beta_j(x,\mu) \int_{-\infty}^{+\infty} \alpha_i(x')\, dx' \int_0^1 \frac{\beta_j(x',\mu')}{\frac{\alpha(x)}{\mu} + \frac{\alpha(x')}{\mu'}} \frac{d\mu'}{\mu'}. \tag{32}$$

Expressions (31) and (32) make it possible: a) with the same choice of the number of terms N in the expansions (4) and (17), to determine the comparative accuracy of the DSA and DRF methods, and b) to find those values of the N_1 and N_2 of the number N at which the DSA and DRF methods provide the same accuracy.

7. Conclusion

The report emphasizes an obvious, but very important physical fact that in the process of multiple scatterings of a quantum in a scattering-absorbing medium, after each successive single act of scattering, the radiation field formed in the medium becomes more and more smooth. The presence of such a property of the resulting radiation field allows us to develop a new view to the solution of the classical, well-studied problem of diffuse reflection. Indeed, in this case, it is more efficient to search for the unknown resulting field of multiple scatterings directly in the decomposed form through a specially constructed system of eigenfunctions (the DRF method) than to derive the final decomposed field form by means of preliminary decomposition of the characteristics of a single act of scattering (the DSA method). That is, the quality of variable separation, in contrast to the methods traditionally used until now (DSA methods), can be directly introduced into the solution of the problem from the very beginning (DRF method), without using any preliminary decomposition or a special representation of the characteristics of the elementary act of scattering. In summary: a) an explicit solution to the problem of diffuse reflection from a semi-infinite medium under isotropic incoherent scattering in the case of the general law of frequency redistribution was presented above by the DRF method, where the variables (x',μ')-frequency and angle of the quantum entering the medium are "separated" from the same quantities (x,μ) of the quantum leaving it, b) a general iterative scheme for numerical calculation of a system of functional nonlinear integral equations and algebraic equations for the joint determination of both eigenfunctions and eigenvalues of the problem is also shown. c) in order to compare the effectiveness of the DSA and DRF methods in an explicit analytical form, some two-way relationships between the desired values of both methods was also obtained .

References

Ambartsumian V. A., 1941, Journal Uchyoniye Zapiski LGU (in Russian), No. 82, Seriya Math. Nauk (Astronomiya), Vipusk 11, Trudi Astron. Observ. LGU, 12, 64

Ambartsumian V. A., 1942, Astron. Zhour. (in Russian), (Engl. transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.159-180, CSP, 2010), 19, 30

Ambartsumian V. A., 1943a, Zhour. Theor. Exp. Phys. (in Russian), (Engl. transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.187-208, CSP , 2010), 13, 313

Ambartsumian V. A., 1943b, Dokl. AN SSSR (in Russian), (Engl. transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.181-185, CSP, 2010), 38, 257

doi: https://doi.org/10.52526/25792776-24.71.1-72

Ambartsumian V. A., 1944a, Izvestiya AN Arm. SSR, Natural Sci. (in Russian), No.1-2, 31, (Engl. transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.209-214, CSP, 2010), p. 31

Ambartsumian V. A., 1944b, J. Phys., 8, 65

Ambartsumian V. A., 1944c, Dokl. AN SSSR (in Russian), (Engl. transl., V. A. Ambartsumian: Selected papers, Part I., Ed. Meylan, pp.203-208, CSP, 2010), 43, 106

Engibaryan N. B., 1971, Astrophysics, 7, 340

Engibaryan N., Nikogosyan A., 1972, Astrophysics, 8, 128

Pikichyan O. V., 1978, Dokl. AN Arm SSR (in Russian), 67, 151

Pikichyan O. V., 1980, Soobshcheniya Byur. Observ. (in Russian), 52, 148

Pikichyan H. V., 2023a, Communications of the Byurakan Astrophysical Observatory, 70, 143

Pikichyan H. V., 2023b, Communications of the Byurakan Astrophysical Observatory, 70, 204

Pikichyan H. V., 2023c, Communications of the Byurakan Astrophysical Observatory, 70, 235

Smokity O., Anikonov A., 2008, Light scattering in media of high optical thickness. St. Petersburg, Nauka Publ

Sobolev V. V., 1963, A Treatise on Radiative Transfer. D. Van Nostrand

doi: https://doi.org/10.52526/25792776-24.71.1-72