An Integral Approach to the Theory of Classical Polytropes

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Abstract

A nonlinear Volterra integral equation of the second kind is used instead of conventional Lane-Emden differential equation to represent an alternative approach to finding exact solutions and analytical approximations to solutions of the Lane-Emden equation for classical polytropic models. This approach enables us to reproduce the well - known Lane- Emden (or, just Emden) functions for polytropic indices n=0,1,5 directly or by making use of the Laplace transform, and, being combined with some heuristic reasonings, derive analytical approximations to exact solutions for n = 1.5, 2 and ∞ in closed forms. The proximity of all suggested analytical approximations to the exact solutions are evaluated with the use of the mean square error estimator. Standard deviations are found to be of 10^{-3} by the order of magnitude. The approximating function of the isothermal density distribution enables us to calculate a theoretical rotation curve that reproduces main features of rotation curves of a set of spiral galaxies. Detailed mathematical calculations will be introduced in an extended paper which is under preparation.

Keywords:*polytropes – polytropic index-Lane-Emden equation-Emden function; Volterra non-linear integral equation – Laplace transform - analytical approximations- dimensionless density distributions*

1. Introduction

Polytropes are self-gravitating gas systems characterized by a specific relationship between the gas pressure (P) and its density (ϱ): $P = K\varrho^{1+\frac{1}{n}}$, where n is the polytropic index and K is the coefficient of proportionality, dependent on the central density and the central velocity dispersion (or the central value of the gravitational potential). Polytropes have been intensively used in the study of stellar structure, theory of stellar evolution (Chandrasekhar (1939), Horedt (2004) (see references therein)), studies of the structure of globular clusters, clusters of galaxies and dark matter distribution. The mathematical foundation of the theory of classical polytropes is the well-known Lane – Emden differential equation, the solutions of which is called Lane-Emden (or, just Emden) function, derived from Poisson equation and the equation of hydrostatic equilibrium, in which the polytropic relation mentioned above is used. The Lane-Emden differential equation can be written in the standard or alternative form (Ivanov, 2018):

$$\frac{1}{\xi^2}\frac{\partial}{\partial}(\xi^2\frac{\partial\theta}{\partial\xi}) = -\theta^n \quad or \quad \frac{d^2(\xi\theta)}{d\xi^2} = -\xi\theta^n \tag{1}$$

where $\theta(\xi)=\frac{\varphi}{\varphi_c}$ dimensionless potential Emden (also Lane-Emden) function, that is the gravitational potential φ, divided by its central value), $\xi = \frac{r}{r_0}$ (r – distance from the center, r_0 - the characteristic radius in a polytropic model) and n as a polytropic index, supplied by the boundary conditions $\theta(0)$=1, $\theta'(0) = 0$. The equation has exact solutions only for n=0,1,5. For all other polytropic models their analytical solutions in closed form do not exist Medvedev & Rybicki (2001) and need to be found via numerical integration. Despite methods of numerical integration have achieved very high level of precision (Horedt (1986), Horedt (2004)) in computing Emden functions the analytical approach is still in demand (see Mach (2012), Milgrom (2021), Motsa & Shateyi (2012)) and references therein), may have a direct relation to physical aspects of the problem Horedt (1987), cast new light on old problems and reduce time for numerical integration. The main goal in this paper is to introduce an integral approach to the theory of classical polytropes as an alternative view on the well-known problem which can be used as an additional tool to the conventional Lane-Emden equation.

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2. Volterra non-linear integral equation of the second kind

The Lane-Emden equation (1) with boundary conditions states the Cauchy problem in mathematical physics, the solution of which can be written in the form of Volterra non-linear integral equation of the second kind:

$$\theta(\xi) = 1 - \int_0^\xi \theta^n(x)x(1 - \frac{x}{\xi})dx \tag{2}$$

for all n $< \infty$. It is required that $\theta(\xi_0) = 0$, ξ_0 - radius of the polytrope which is finite for n < 5 and infinite for n ≥ 5 . Only the functions satisfying all the conditions mentioned above are thought to be physically acceptable. In case of an isothermal sphere (n $= \infty$) the corresponding differential equation can be written in the form $d^2(\xi\psi)/d\xi^2 = \xi e^{-\psi(\xi)}$, supplied by the boundary conditions $\psi(0) = 0$, $\psi'(0) = 0$, and can be converted into the following Volterra homogenous non-linear integral equation of the second kind

$$\psi(\xi) = \int_0^\xi e^{-\psi(\xi)}x(1 - \frac{x}{\psi})dx \tag{3}$$

The exponential term in (3) represents the dimensionless density distribution in the isothermal sphere $\rho(\xi) = exp(-\psi(\xi))$.
It is obvious that all polytropic solutions are symmetric $\theta(\xi) = \theta(-\xi)$ and their expansions into power series or Pade approximations must include only even powers of ξ (see, for example, **??**. In addition to this we can state that for a large domain of polytropic indices density distributions should have an inflection point. This may be incorrect for some polytropic models (such as $0 \leq$ n < 1).
The integral equations presented above were derived in Saiyan (1997) (not published), then by Shaudt (2000), who considered the existence and uniqueness as well as regularity of the general Lane-Emden solution within the fixed-point problem. Horedt (2004) refers to this result in his book and shows how the equations can be derived from general considerations by making use of Green's formula.

3. Exact solutions

3.1. n=0

This is a model of incompressible liquid ($\varrho = const$). The solution can be obtained via direct simple integration of (2), which gives well - known function $\theta(\xi) = 1 - \frac{\xi^2}{6}$.

3.2. n=1

The equation (3), after being multiplied by ξ, takes the form

$$\xi\theta(\xi) = \xi - \int_0^\xi \theta(x)x(\xi - x)dx \tag{4}$$

with the convolution term in the righthand integral. This is the standard linear Volterra integral equation of the second kind for the function $\xi\theta(\xi)$ and can easily be solved by applying direct and then inverse Laplace transforms or by Picard approximations. Its solution is $\sin\xi$. Thus, we derived Ritter's solution $\theta(\xi) = \frac{\sin(\xi)}{\xi}$, which is the Bessel spherical function $\mathrm{j}_0(\xi)$.

3.3. n=5

This case is more challenging in terms of direct calculations because of extreme non-linearity of the Volterra equation (2). We assume that the functions $\theta(\xi)$ and $\theta^5(\xi)$ themselves are Laplace images of the originals $F(t)t^\nu$ and $\Phi(t)t^\mu$, continuous on $\mathbf{R^1}$ and have appropriate rates of exponential growth

$$\theta(\xi) = C_1 \int_0^\infty F(t)t^\nu e^{-\xi t/\alpha}dt \qquad \theta^5(\xi) = C_2 \int_0^\infty \Phi(t)t^\mu e^{-\xi t/\alpha}dt \tag{5}$$

Here C_1 and C_2 are constants to be chosen and α is a free parameter to be defined. Taking derivatives by ξ in (6) and substituting them in the Lane-Emden equation, one can show that if functions $F(t)$ and $\Phi(t)$ are Bessel J- functions $F(t) = J_0(t)$, $\Phi(t) = J_2(t)$ and $C_1 = C_2\alpha^2 = 1$ for simplicity, as well as ν=0, $\mu = \frac{n-1}{2} = 2$, then the Lane -Emden equation turns into the following recurrence relation for Bessel functions

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$tJ_1(t) = 2J_2(t) + tJ_2'(t)$. Thus, we have the following representations for $\theta(\xi)$ and $\theta^5(\xi)$ (Bateman & Erdelyi, 1974):

$$\theta(\xi) = \int_0^\infty e^{-\xi t/\alpha} J_0(t) t^\nu dt = \frac{1}{\sqrt{1+\xi^2/\alpha^2}}, \qquad \theta^5(\xi) = \frac{1}{\alpha^2}\int_0^\infty e^{-\xi t/\alpha} J_2(t) t^2 dt = \frac{3}{\alpha^2(1+\xi^2/\alpha^2)^{\frac{5}{2}}} \tag{6}$$

By comparing last integrals, we see that one can make (8) 5th power of (7) if $\alpha = \sqrt{3}$. Thus, we arrive at the following final representation $\theta(\xi) = \frac{1}{\sqrt{1+\xi^2/3}}$ and $\theta^5(\xi) = \frac{1}{(1+\xi^2/3)^{5/2}}$. This is the Emden-Schuster solution for n=5 polytrope with the finite mass, but infinite radius. In the meantime, the formula describes Plummer- Schuster model of globular clusters.

4. Approximate solutions

4.1. n=1.5 and n=2

As we know the case n=1.5 describes internal structure of white dwarfs of low mass with degenerate nonrelativistic electronic gas, fully convective star cores, and the quasi-stationary stellar systems with local thermodynamic equilibrium (Gurzadyan & Kechek (1979)). The case n=2 is within the range acceptable for common stars ($1.5 \leq n \leq 3$). The integral equations of these models are complex, but their linear approximation encourage us to search for possible solutions in the form close to $j_0(\xi/\sqrt{n})$, where $j_0(x)$ is a spherical Bessel function. By multiplying this function by correction term in the form of quadratic expression $\theta(\xi){=}j_0(\xi/\sqrt{n})(1-\frac{\xi^2}{\xi_1^2})$ for n=1.5 and varying the free parameter ξ_1 we could find very good approximations for the Emden function of the polytrope. For n =1.5 ξ_1= 4.47456182 (the radius of the polytrope is found to be 3.8476, the real radius is 3.6538). For n=2 the correction factor is used in the form $1/[1+\xi^2/\xi_1^2]$ and gives $\xi_1 = 3.6$ (the radius obtained is 4.443 against the real 4.353). Graphs of the approximations to Emden functions are shown on Fig.1 and compared with results of numerical integration of the Lane – Emden equation Horedt (1986). The standard deviation of the suggested functions from exact solution is $8.087{\cdot}10^{-3}$ for n=1.5 and $4.12321{\cdot}10^{-3}$ for n=2.

4.2. Isothermal model (n=∞)

This model is of particular interest in astrophysics and stellar dynamics and has been studied in detail by Chandrasekhar (1939). Equation (3) gives us another opportunity to study some basic features of the model, such as the density distribution near the center of isothermal spheres and the main term of its asymptotic behavior. For $\xi \ll 1$ the exponential term in (3) can be replaced by 1 as a first approximation and we obtain $\psi(\xi) = \frac{\xi^2}{6}$ Thus, the density distribution near the center is $\rho(\xi) \approx exp(-\frac{\xi^2}{6}) \approx 1 - \frac{\xi^2}{6}$, which slightly differs from the density of incompressible liquid. The analysis of the equation (3) for the asymptotic behavior of the dimensionless potential and density distribution boils down to the simple equation:

$$\psi(\xi) = \int_0^\xi e^{-\psi(x)} x dx \tag{7}$$

that contains Emden's main asymptotic term $\psi(\xi) = 2ln\xi - ln2$ Chandrasekhar (1939), Ivanov (2018) which gives $e^{-\psi(\xi)}{\sim}\frac{2}{\xi^2}$ for the density distribution (solution for a singular isothermal sphere). Because the exact non-singular solution of the equation (6) is unknown, we can find its acceptable approximation by making use of heuristic reasoning and knowing the behavior of the distribution density in the vicinity of the center of the isothermal model and, also, its asymptotic behavior. Here we assume that in the most general form, for $n \geq 5$, polytropic models with infinite radii can be described by functions of the type $\theta(\xi)= \frac{1}{(1+\delta x^2)^\beta}$, where values of the constants δ his gives δ=1/2 and β= 1/3 at $\xi{\to}0$ and β=1 at $\xi{\to}\infty$. In other words, we accept the possibility that β is a function of the dimensionless distance, which on the entire real axis varies within short range from 1/3 to 1. From the symmetry requirement of the function $\theta(\xi)$ it follows that in the simplest case it must depend on ξ^2. We have tested a set of functions with this property and have found that the best approximation for $\beta(\xi)$ (according to the mean square error estimator) is

$$\beta(\xi) = \frac{1+\frac{\xi^2}{4}}{3+\frac{\xi^2}{4}} \tag{8}$$

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which results in the following expression for the density distribution:

$$e^{-\psi(\xi)} \approx \varrho(\xi) = (1+\frac{\xi^2}{2})^{-\beta(\xi)} \tag{9}$$

The mean square deviation of its graph from the exact density distribution for the isothermal sphere, obtained by numerical integration, is $2.685 \cdot 10^{-3}$ (Fig.1). The dimensionless potential $\psi(\xi)$ and the acceleration (a(ξ)) in this case are determined by the following formulas:

$$\psi(\xi) = \frac{1+\frac{\xi^2}{4}}{3+\frac{\xi^2}{4}} ln(1+\frac{\xi^2}{2}) \qquad \alpha(\xi) = \frac{\xi}{3+\frac{\xi^2}{4}}[\frac{1+\frac{\xi^2}{4}}{1+\frac{\xi^2}{2}} + \frac{ln(1+\frac{\xi^2}{2})}{3+\frac{\xi^2}{4}}] \tag{10}$$

At large distances ($\xi \to \infty$) the acceleration decreases inversely proportional to the distance: a(ξ) $\sim \frac{1}{\xi}$. For a circular rotational motion then its orbital velocity at large distances reaches at saturation level $v^2 \sim \xi a(\xi) =$ const, while at short distances the velocity changes linearly $v \sim \xi$ wiith the distance from the center, which is in compliance with the properties of the rotation curves of spiral galaxies (see, for example, Wojnar et al. (2018) and references therein). The fact that centers of those galaxies rotate as a rigid body implies that they can be approximately described by the model of an incompressible liquid, which is typical for the central regions of all polytropic models. More detailed discussions of the results obtained as well as comparison with observational data and other theoretical rotation curves will be presented in a later paper.

5. Discussion

The Volterra nonlinear integral equation of the second kind is suggested as an alternative approach to the theory of classical polytropic models in finding exact solutions and analytical approximations to Emden functions that have been obtained so far via numerical integration. The equation is invariant with respect to homology transform. The approach reproduces main solutions of the classical polytropic theory (except for singular solutions which are disallowed by Volterra equation (2)) and helps in finding analytical approximations in closed forms to polytropic models with n=1.5,2 and isothermal spheres. The calculated theoretical rotation curves in the framework of an isothermal model reproduce main features of typical rotation curves of spiral galaxies (Fig 2).

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References

Bateman H., Erdelyi A., 1974, Higher Transcedental Functions.II. Nauka, Moscow,Russia

Chandrasekhar S. R., 1939, An Introduction to the Study of StellarStructure. The University of Chicago Press, Chicago, Illiinois, USA

Gurzadyan V. G., Kechek A. G., 1979, Preprint, Lebedev Institute AN SSSRl, 180, 13

Horedt G. P., 1986, Astrophys.Space.Sci., 126, 357

Horedt G. P., 1987, A&A, 172, 359

Horedt G. P., 2004, Polytropes-Applications in Astrophysics and related fields. Kluwer Academic Publisher, Dordrecht, Netherlands

Ivanov V., 2018, Astrophysics of stars. Sant-Petersburg State University, Sant-Petersburg, Russia

Mach P., 2012, J.Math.Phys, 53, 1

Medvedev M. E., Rybicki G., 2001, ApJ, 555, 863

Milgrom M., 2021, Phys.Rev.D,, 103, 1

Motsa S. S., Shateyi S., 2012, Mathematical Problems in engineeringl, 2012, 1

Saiyan G. A., 1997, PhD Disertation. National Academy of Sciences of the Republic of Armenia, Yerevan, Armenia

Shaudt U. M., 2000, Ann.Henri Poincarel, 1, 945

Wojnar A., Sporea C. A., A. B., 2018, Galaxiesl, 6, 13

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Appendices

Appendix A Graphs of analytical approximations in closed forms

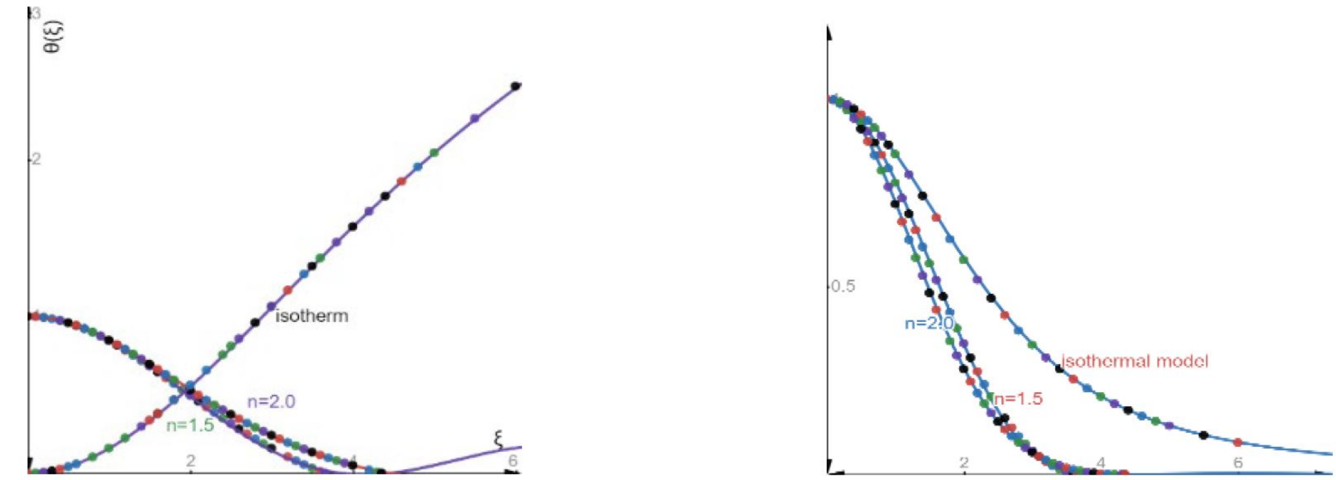

Figure 1. Continuous curves represent analytical approximations of Emden functions (left-hand graph) and density distributions for polytropic models with n = 1.5,2.0,∞. Dots represent exact solutions, obtained by Horedt (1986) via numerical integration.

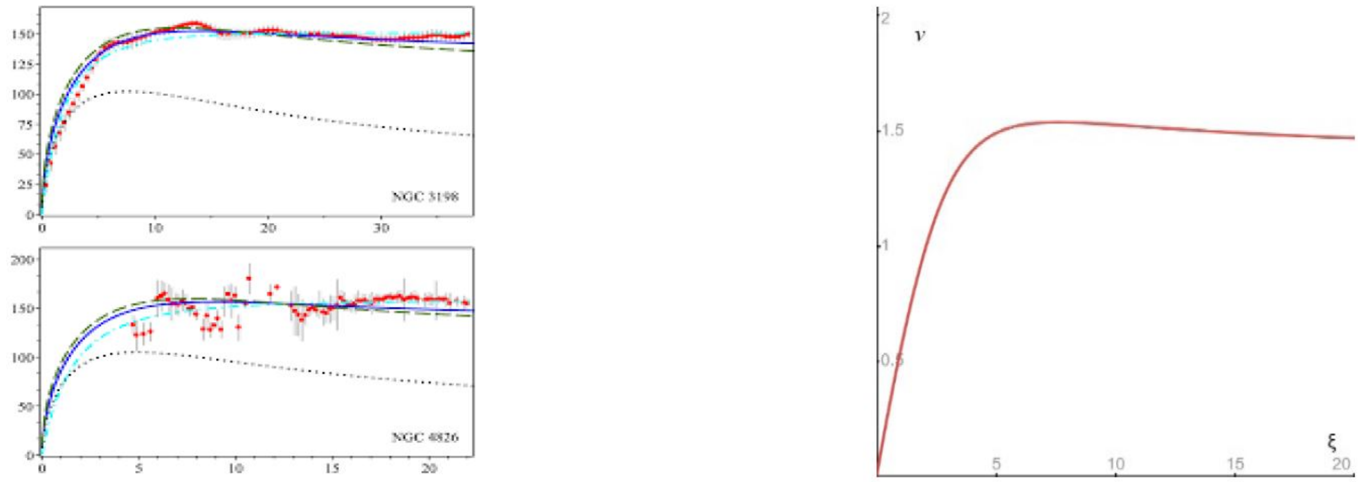

Figure 2. The rotation curves of spiral galaxies (NGC 3198, NGC 4826 from Wojnar et al. (2018) (left-hand graph) and the theoretical rotation curve for an isothermal sphere with the dimensionless gravitational potential in (10).

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