Quantum interference of a de Broglie wave of a Dirac particle beyond the `hypothesis of locality´. Part III: Geometry

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Abstract

This is the last of three articles that explore the quantum mechanical inertial properties of the Dirac particle beyond the `hypothesis of locality´. This is done within the framework of the *Master Space*-Teleparallel Supergravity ($\widetilde{MS}_p$-TSG) (Ter-Kazarian, 2025a) theory, which we recently proposed to account for inertial effects (Ter-Kazarian, 2026). In present article, we review the technical details of geometry beyond the `hypothesis of locality´, referred to the 4D background Minkowski space in noninertial frame of arbitrary accelerating and rotating observer (Ter-Kazarian, 2025b). Given the anholonomic frame and coframe members, the object of anholonomicity and connection (Ter-Kazarian, 2025b), we compute the connection 1-forms, the curvature 2-form and write it in terms of Riemann curvature tensor. Then we derive the Riemann tensor in an anholonomic frame and compute the Riemann tensor, Ricci tensor, Ricci scalar, Kretschmann scalar.

Keywords: *Teleparallel Supergravity–Spacetime Deformation–Inertia Effects–Quantum interference*

1. Introduction

The experiments of quantum interference of De Broglie matter waves of Dirac particle are reviewed by (Abele & Leeb, 2012, Atwood & et al., 1984, Bonse & Wroblewski, 1983, Colella et al., 1975, Hasegawa & Rauch, 2011, Kajari et al., 2010, Michelson et al., 1925, Page, 1975, Rauch & Werner, 2000, Staudenmann et al., 1980). In the meantime, the theoretical studies of the relativistic quantum theory in a curved spacetime have predicted a number of interesting manifestations of the spin-gravity coupling for a Dirac particle, see e.g. (Audretsch & Schafer, 1978, Cai & Papini, 1991, 1992, Fischbach et al., 1981, Hehl & Ni, 1990, Obukhov, 2001, 2002, Ryder, 1998, Singh & Papini, 2000, Varjú & Ryder, 1998, 2000, de Oliveira & Tiomno, 1962). For a performing the laboratory measurements, it is necessary to give a theoretical description of the measurements of accelerated observers. This is, usually, done via the `hypothesis of locality´, used to extend Lorentz invariance to accelerated observers within the framework of Special Relativity, see e.g. (Hehl & Ni, 1990, Hehl et al., 1991, Maluf & Faria, 2008, Maluf et al., 2007, Marzlin, 1996, Mashhoon, 2002, 2011, Misner et al., 1973, Synge, 1960) and references therein. However, many scientists found its basic conceptual framework unsatisfactory. In general case, the hypothesis of locality will have to be extended to describe physics for arbitrarily accelerated observers.

In Ter-Kazarian (2025b) (first article of three), we computed the object of anholonomicity and the connection defined with respect to the anholonomic frame, beyond the `hypothesis of locality´. Then we derived the explicit final form of the Dirac equation for an observer in a reference frame that is accelerated with a three-acceleration $\vec{a}$ and rotating with angular frequency $\vec{\omega}$. However, the purely imaginary potential term from the Dirac Hamiltonian is associated with non-Hermitian contributions due to coordinate transformations in accelerated frames. Residual imaginary terms are artifacts.

To eliminate to all orders these terms, in Ter-Kazarian (2025c) (second article of three), we apply the standard techniques used in relativistic quantum mechanics and quantum field theory, where non-Hermitian terms can be removed via suitable similarity transformations. This standard method allows us to choose a physically more suitable reference frame. The expectation values of physical observables remain real.

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No imaginary contamination remains in physical quantities. Thus the energy, momentum, probability, etc. remain real and consistent. Next we investigated the low-energy properties, avoiding solutions with negative energy. In the method employed for reducing the Dirac Hamiltonian to non-relativistic two-component form, in order to decouple the positive and the negative energy states, we use an approximate scheme of the Foldy-Wouthuysen canonical transformation of the Dirac Hamiltonian for a free particle. This is performed by an infinite sequence of FW-transformations leading to a deformed Hamiltonian, which is an infinite series in powers of $(1/m)$. Evaluating the operator products to the desired order of accuracy, we find the deformed, non-relativistic Hamiltonian. We then found the inertial effects for a massive Dirac fermion in non-relativistic approximation, which are displayed beyond the `hypothesis of locality´ as extended (deformed) versions of the standard effects. The latter are well-known important inertial effects such as the redshift effect (Colella-Overhauser-Werner experiment), the Sagnac-type effect, the spin rotation effect (Mashhoon), the kinetic energy redshift effect, the new inertial spin-orbit coupling. Expanding further the deformation coefficients, several new effects will rather appeared involving spin, angular momentum, proper linear 3-acceleration $\vec{a}$ and proper 3-angular velocity $\vec{\omega}$ in various mixed combinations.

To complete this stage of investigation of quantum interference of a de Broglie wave of a Dirac particle beyond the `hypothesis of locality´, in present article, we review and clarify the technical details of geometry beyond the `hypothesis of locality´, referred to the 4D background Minkowski space in noninertial frame of arbitrary accelerating and rotating observer. Given the anholonomic frame and coframe members, the object of anholonomicity and connection (Ter-Kazarian, 2025b), we compute the connection 1-forms, the curvature 2-forms and write it in terms of Riemann curvature tensor. Then we derive the Riemann tensor in an anholonomic frame and compute the Riemann tensor, Ricci tensor, Ricci scalar, Kretschmann scalar.

We proceed according to the following structure. To start with, in section 2 we briefly review the orthonormal frame, the object of anholonomicity and the connections, beyond the `hypothesis of locality´. In section 3 we compute the connection 1-forms $\omega^{(\hat{\mu})}{}_{(\hat{\nu})}$, with verification of the connection 1-forms (subsect. 3.1) and Cartan's first structure equation (subsect. 3.2). On these premises, in section 4 we compute the curvature 2-forms $\Omega^{(\hat{\mu})}{}_{(\hat{\nu})}$, and again from scratch (subsect. 4.1). Derivation of the Riemann tensor in an anholonomic frame is presented in section 5. In section 6 we compute the Riemann tensor. In section 7 we compute the Ricci tensor (subsect. 7.1), Ricci scalar (subsect. 7.2), Kretschmann scalar (subsect. 7.3). As concluding remarks, in section 8 , we review the key points of this report. It is worthwhile to recall some technical details collected in Appendix. Unless indicated otherwise, the natural units, $h = c = 1$ are used throughout.

2. The orthonormal frame, the object of anholonomicity and the connections

To make this article understandable, the interested reader is referred to the original papers (Ter-Kazarian, 2024a, 2025a,b, 2026) (see also (Ter-Kazarian, 2024b,c,d)). In this section, we briefly recall some preliminary geometrical structures used in (Ter-Kazarian, 2025b).

We consider only mass points, then the non-inertial frame of reference in the Minkowski space of SR is represented by a curvilinear coordinate system, since it is conventionally accepted to use the names `curvilinear coordinate system´ and `non-inertial system´ interchangeably. Consider the accelerated motion of a relativistic test particle in Minkowski 4D background flat space, M_4, under the unbalanced net force other than gravitational. The hypothesis of locality assumes the equivalence of an accelerated observer and an instantaneously moving inertial observer, i.e. it links the measurements of the accelerated observer with the measurements of the inertial observer (see Appendix/(1)-(3)). This immediately leads to the startling view within the framework of the $\widetilde{MS}_p$-TSG theory, of replacing the non-inertial reference frame $S^{(\varrho)}_{(2)}$, which is held stationary in the deformed master space $\underline{V}^{(\varrho)}_2$ $(\varrho \neq 0)$, with a continuous infinity set of the inertial frames $\{S^{(0)}_{(2)}, S'^{(0)}_{(2)}, S''^{(0)}_{(2)}, ...\}$ given in $\underline{V}^{(\varrho)}_2$ $(\varrho = 0)$. In other words, the hypothesis of locality leads to the 2D semi-Riemannian space, $V^{(0)}_2(\varrho = 0)$, with the incomplete metric of $\widetilde{g}$ (see (69)). Here $\varrho(\underline{x})$ is the *local rate* of instantaneously change of a constant velocity (both magnitude and direction) of a massive particle in 4D Minkowski space under the unbalanced net force. Namely, this assumption replaces the space $\widetilde{MS}_p \equiv \underline{V}^{(\varrho)}_2$ with the $\underline{V}^{(0)}_2$. Therefore, our further strategy is to consider the two-steps deformation

$$\Omega(\varrho) : \underline{M}_2 \to \underline{V}^{(\varrho)}_2, \tag{1}$$

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which is composed of the two deformations as follows:

$$\Omega : \underline{M}_2 \to \underline{V}_2^{(0)}, \quad \widetilde{\Omega}(\varrho) : \underline{V}_2^{(0)} \to \underline{V}_2^{(\varrho)}, \tag{2}$$

where the *world-deformation* tensors $\Omega(\varrho)$ and $\widetilde{\Omega}(\varrho)$ are functions of $\varrho(\underline{x})$. It follows that the components of metric tensor in $\underline{V}_2^{(\varrho)}$ read

$$\begin{array}{l} g_{\tilde{0}\tilde{0}} = (1 + \frac{\varrho v^{\underline{1}}}{\sqrt{2}})^2 - \frac{\varrho^2}{2}, \quad g_{\tilde{1}\tilde{1}} = -(1 - \frac{\varrho v^{\underline{1}}}{\sqrt{2}})^2 + \frac{\varrho^2}{2}, \\ g_{\tilde{1}\tilde{0}} = g_{\tilde{0}\tilde{1}} = -\sqrt{2}\varrho. \end{array} \tag{3}$$

Using the following embedding relations as a converting guide (see App./(2),(3)):

$$\begin{array}{l} g_{\tilde{0}\tilde{0}}(d\underline{X}^{\underline{0}})^2 = g_{\tilde{0}\tilde{0}}(dX^0)^2, \quad g_{\tilde{1}\tilde{1}}(d\underline{X}^{\underline{1}})^2 \\ = g_{\tilde{1}\tilde{1}}(d\vec{X} \cdot d\vec{X}), \quad g_{\tilde{1}\tilde{0}} d\underline{X}^{\underline{1}} = g_{\tilde{i}\tilde{0}} d\underline{X}^{\underline{i}}, \\ g_{\tilde{i}\tilde{0}} = g_{\tilde{0}\tilde{i}} = n_i g_{\tilde{0}\tilde{1}} = n_i g_{\tilde{1}\tilde{0}}, \quad b_{2i} = n_i b_2, \end{array} \tag{4}$$

by means of (70), (71), we obtain the generalized frame and coframe members referred to the 4D background space as follows:

$$\begin{array}{l} e_{(\hat{0})} = b_0^{-1} \left\{ (1 + \vec{a} \cdot \vec{X})\, e_{\hat{0}} + (\frac{b_{2i}}{b_1} + (\vec{\omega} \times \vec{X})^i)\, e_{\hat{i}} \right\}, \\ e_{(\hat{i})} = b_1^{-1}\, e_{\hat{i}}, \end{array} \tag{5}$$

and

$$\begin{array}{l} \vartheta^{(\hat{0})} = \frac{b_0}{1 + \vec{a} \cdot \vec{X}}\, \vartheta^{\hat{0}}, \\ \vartheta^{(\hat{i})} = b_1\, \vartheta^{\hat{i}} - \frac{1}{1 + \vec{a} \cdot \vec{X}}\, [b_{2i} + b_1 (\vec{\omega} \times \vec{X})^i]\, \vartheta^{\hat{0}}, \end{array} \tag{6}$$

provided,

$$\begin{array}{l} b_1(\varrho) \equiv (-g_{\tilde{1}\tilde{1}})^{1/2}, \quad b_2(\varrho) = \frac{g_{\tilde{1}\tilde{0}} + g_{\tilde{0}\tilde{1}}}{2(-g_{\tilde{1}\tilde{1}})^{1/2}(\varrho)}, \\ b_0(\varrho) = (g_{\tilde{0}\tilde{0}} + b_2(\varrho)^2)^{1/2}, \quad \varrho(\tilde{s}) = \sqrt{2} \int_0^{\tilde{s}} |\vec{a} \wedge \vec{u} + \vec{\omega} \times \vec{u}| d\tilde{s}'. \end{array} \tag{7}$$

Whereas the orthonormal frame $e_{\hat{a}}$, can be written with respect to curvilinear or Cartesian coordinates (54). The coframe members $\vartheta^{\hat{b}}$ are the objects of dual counterpart: $e_{\hat{a}} \rfloor \vartheta^{\hat{b}} = \delta^{\hat{b}}_{\hat{a}}$. The components of the field of the orthonormal frame and the components of the field of the dual coreframe are respectively given by (59) and (60). Similarly, by means of (5) and (6), the orthonormal frame $e_{(\hat{a})}(\varrho)$ and coframe $\vartheta^{(\hat{b})}(\varrho)$, carried by an accelerated observer, can be rewritten with respect to curvilinear coordinates:

$$e_{(\hat{a})}(\varrho) = e_{(\hat{a})}{}^{\mu}(\varrho)\, e_{\mu}, \quad \vartheta^{(\hat{b})}(\varrho) = e^{(\hat{b})}{}_{\nu}(\varrho)\, \vartheta^{\nu}.$$

Then

$$e_{(\hat{0})} = b_0^{-1}(e_0 + \frac{b_2^i}{b_1} e_i), \quad e_{(\hat{i})} = b_0^{-1} e_i, \tag{8}$$

and

$$\vartheta^{(\hat{0})} = b_0 \vartheta^0, \quad \vartheta^{(\hat{i})} = b_1 \left(\vartheta^i - \frac{b_2^i}{b_1} \vartheta^0 \right)), \tag{9}$$

with the components of the orthonormal frame field and their reciprocals

$$\begin{array}{l} e_{(\hat{0})}{}^{0}(\varrho) = b_0^{-1}, \quad e_{(\hat{0})}{}^{i}(\varrho) = (b_0 b_1)^{-1} b_2^i, \\ e_{(\hat{i})}{}^{0}(\varrho) = 0, \quad e_{(\hat{i})}{}^{j}(\varrho) = b_1^{-1}(\varrho)\, \delta^j_i, \end{array} \tag{10}$$

and

$$\begin{array}{l} e^{(\hat{0})}{}_{0}(\varrho) = b_0, \quad e^{(\hat{0})}{}_{i}(\varrho) = 0, \\ e^{(\hat{i})}{}_{0}(\varrho) = -b_{2i}, \quad e^{(\hat{i})}{}_{j}(\varrho) = b_1\, \delta^i_j, \end{array} \tag{11}$$

respectively. The complete metric in noninertial frame of arbitrary accelerating and rotating observer in Minkowski spacetime reads

$$\begin{array}{l} d\tilde{s}^2(\varrho) = g_{\mu\nu}(\varrho) dX^{\mu} dX^{\nu} = \vartheta^{(\hat{0})} \otimes \vartheta^{(\hat{0})} - \vartheta^{(\hat{i})} \otimes \vartheta^{(\hat{i})} \\ = (dX^0)^2 \left[b_0^2 - b_2^2 - (2b_1^2 + 1)(\vec{\omega} \cdot \vec{X})^2 - (2b_1 + 1)\vec{b}_2 \cdot (\vec{\omega} \cdot \vec{X}) \right] - b_1^2\, d\vec{X} \cdot d\vec{X} \\ - dX^0\, d\vec{X} \cdot \left[b_1 (2b_1 + 1)(\vec{\omega} \times \vec{X}) + \vec{b}_2 \right]. \end{array} \tag{12}$$

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Thus, we are given the anholonomic frame and coframe members with respect to curvilinear coordinates, beyond the `hypothesis of locality´, referred to the 4D background Minkowski space in noninertial frame of arbitrary accelerating and rotating observer (Ter-Kazarian, 2025b):

$$e_{(\hat{0})} = b_0^{-1}\left(e_0 - \bar{v}^k e_k\right) = b_0^{-1}\left(\partial_0 - \bar{v}^k \partial_k\right), \quad e_{(\hat{i})} = b_1^{-1} e_i = b_1^{-1}\partial_i, \tag{13}$$

and

$$\vartheta^{(\hat{0})} = b_0 \vartheta^0, \quad \vartheta^{(\hat{i})} = b_1\left(\vartheta^i + b_4 v^i \vartheta^0\right)), \tag{14}$$

where we denote $\frac{b_2^j(\varrho)}{b_1(\varrho)} \equiv -b_4(\varrho)v^j, \quad v^j = (\vec{\omega} \times \vec{X})^j$.

The components of anholonomicity (the structure-constants) read

$$C^{(\hat{\lambda})}{}_{(\hat{\mu})(\hat{\nu})} = e_{(\hat{\mu})}{}^{\mu} e_{(\hat{\nu})}{}^{\nu}(\partial_\mu e^{(\hat{\lambda})}{}_\nu - \partial_\nu e^{(\hat{\lambda})}{}_\mu). \tag{15}$$

To lower the upper index by a metric $o_{(\hat{\rho})(\hat{\lambda})}$, using an orthogonal basis $o = (diag{+1}, -1, -1, -1)$, the structure-constants become $C_{(\hat{\mu})(\hat{\nu})(\hat{\rho})} = o_{(\hat{\rho})(\hat{\lambda})} C^{(\hat{\lambda})}{}_{(\hat{\mu})(\hat{\nu})}$. Hence, by virtue of (13) and (14), we have to calculate all non vanishing components $C_{(\hat{\mu})(\hat{\nu})(\hat{\lambda})}$ of the anholonomicity (Ter-Kazarian, 2025b):

$$\begin{array}{l} C_{(\hat{0})(\hat{i})(\hat{0})} = -C_{(\hat{i})(\hat{0})(\hat{0})} = \frac{1}{b_1^{-1}}\partial_i \ln b_0, C_{(\hat{0})(\hat{i})(\hat{j})} = -C_{(\hat{i})(\hat{0})(\hat{j})} = -\frac{\partial_i v_j}{b_0} + \vec{v}\cdot(\nabla b_1^{-1})\delta_{ij}, \\ C_{(\hat{i})(\hat{j})(\hat{k})} = -C_{(\hat{j})(\hat{i})(\hat{k})} = -(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}, \quad C_{(\hat{i})(\hat{j})(\hat{0})} = C_{(\hat{\mu})(\hat{\nu})(\hat{\lambda})} = 0. \end{array} \tag{16}$$

Using (16), the connection components read (Ter-Kazarian, 2025b)

$$\begin{array}{l} \Gamma_{(\hat{0})(\hat{i})(\hat{0})} = -\Gamma_{(\hat{i})(\hat{0})(\hat{0})} = \frac{1}{b_1^{-1}}\partial_i \ln b_0, \quad \Gamma_{(\hat{i})(\hat{j})(\hat{0})} = \frac{1}{2b_0}(\partial_i \bar{v}_j - \partial_j \bar{v}_i) = \frac{b_4}{b_0}\partial_i v_j + \frac{1}{2b_0}\left[(\partial_i b_4)v_j - (\partial_j b_4)v_i\right], \\ \Gamma_{(\hat{i})(\hat{j})(\hat{k})} = -(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}, \quad \Gamma_{(\hat{0})(\hat{i})(\hat{j})} = -\Gamma_{(\hat{i})(\hat{0})(\hat{j})} = -\frac{1}{2b_0}(\partial_i \bar{v}_j + \partial_j \bar{v}_i) \\ +\vec{v}\cdot(\nabla b_1^{-1})\delta_{ij} = -\frac{1}{2b_0}\left[(\partial_i b_4)v_j + (\partial_j b_4)v_i\right] + \vec{v}\cdot(\nabla b_1^{-1})\delta_{ij}, \quad \Gamma_{(\hat{0})(\hat{0})(\hat{0})} = \Gamma_{(\hat{\mu})(\hat{\nu})(\hat{i})} = 0. \end{array} \tag{17}$$

In next two sections, by means of (16) and (17), we have to calculate respectively the connection 1-forms $\omega^{(\hat{\mu})}{}_{(\hat{\nu})}$ and the curvature 2-forms $\Omega^{(\hat{\mu})}{}_{(\hat{\nu})}$. We will use Cartan's two structure equations.

3. The Connection 1-forms, $\omega^{(\hat{\mu})}{}_{(\hat{\nu})}$

Below we bring the explicit computation of the connection 1-forms $\omega^{(\hat{\mu})}{}_{(\hat{\nu})}$ for the anholonomic tetrad $e_{(\hat{\mu})}$ (13), using Cartan's first structure equation:

$$d\theta^{(\hat{\mu})} + \omega^{(\hat{\mu})}{}_{(\hat{\nu})} \wedge \theta^{(\hat{\nu})} = 0, \qquad \omega_{(\hat{\mu})(\hat{\nu})} = -\omega_{(\hat{\nu})(\hat{\mu})},$$

and the relation between connection 1-forms and the Ricci rotation coefficients:

$$\omega_{(\hat{\mu})(\hat{\nu})} = \Gamma_{(\hat{\mu})(\hat{\nu})(\hat{\rho})}\, \theta^{(\hat{\rho})}. \tag{18}$$

The dual coframe $\theta^{(\hat{\mu})}$ (14) reads

$$\theta^{(\hat{0})} = b_0\, dX^0, \qquad \theta^{(\hat{i})} = b_1\,(dX^i + \bar{v}^i\, dX^0). \tag{19}$$

Thus the connection 1-forms are obtained simply by inserting the corresponding coefficients. We now write every nonvanishing $\omega^{(\hat{\mu})}{}_{(\hat{\nu})}$ explicitly.

$$\omega^{(\hat{0})}{}_{(\hat{i})} = \Gamma_{(\hat{0})(\hat{i})(\hat{0})}\, \theta^{(\hat{0})} + \Gamma_{(\hat{0})(\hat{i})(\hat{j})}\, \theta^{(\hat{j})}.$$

Insert coefficients. (a) Time–space components (17):

$$\Gamma_{(\hat{0})(\hat{i})(\hat{0})} = b_1\, \partial_i \ln b_0, \quad \Gamma_{(\hat{0})(\hat{i})(\hat{0})}\theta^{(\hat{0})} = b_0 b_1 (\partial_i \ln b_0)\, dX^0.$$

(b) Mixed spatial pieces:

$$\Gamma_{(\hat{0})(\hat{i})(\hat{j})} = -\frac{1}{2b_0}(\partial_i \bar{v}_j + \partial_j \bar{v}_i) + (\bar{v}\cdot\nabla b_1^{-1})\delta_{ij},$$

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$$\Gamma_{(\hat{0})(\hat{i})(\hat{j})}\theta^{(\hat{j})} = \Big[-\frac{1}{2b_0}(\partial_i\bar{v}_j + \partial_j\bar{v}_i) + (\bar{v}\cdot\nabla b_1^{-1})\delta_{ij}\Big] b_1(dX^j + \bar{v}^j dX^0).$$

Final expression becomes

$$\omega^{(\hat{0})}{}_{(\hat{i})} = b_0 b_1(\partial_i \ln b_0)\, dX^0 + b_1\Big[-\tfrac{1}{2b_0}(\partial_i\bar{v}_j + \partial_j\bar{v}_i) + (\bar{v}\cdot\nabla b_1^{-1})\delta_{ij}\Big](dX^j + \bar{v}^j dX^0), \tag{20}$$

and by antisymmetry $\omega^{(\hat{i})}{}_{(\hat{0})} = -\omega^{(\hat{0})}{}_{(\hat{i})}$. The purely spatial components $\omega^{(\hat{i})}{}_{(\hat{j})}$ are written

$$\omega^{(\hat{i})}{}_{(\hat{j})} = \Gamma_{(\hat{i})(\hat{j})(\hat{k})}\,\theta^{(\hat{k})} + \Gamma_{(\hat{i})(\hat{j})(\hat{0})}\,\theta^{(\hat{0})}.$$

Insert coefficients. (a) Spatial pieces:

$$\Gamma_{(\hat{i})(\hat{j})(\hat{k})} = -(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}, \quad \Gamma_{(\hat{i})(\hat{j})(\hat{k})}\theta^{(\hat{k})} = \big[-(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}\big] b_1(dX^k + \bar{v}^k dX^0).$$

(b) Rotational (vorticity) terms:

$$\Gamma_{(\hat{i})(\hat{j})(\hat{0})} = \frac{1}{2b_0}(\partial_i\bar{v}_j - \partial_j\bar{v}_i), \quad \Gamma_{(\hat{i})(\hat{j})(\hat{0})}\theta^{(\hat{0})} = \frac{1}{2}(\partial_i\bar{v}_j - \partial_j\bar{v}_i)\, dX^0.$$

Final expression reads

$$\omega^{(\hat{i})}{}_{(\hat{j})} = \big[-(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}\big] b_1(dX^k + \bar{v}^k dX^0) + \tfrac{1}{2}(\partial_i\bar{v}_j - \partial_j\bar{v}_i)\, dX^0 \tag{21}$$

Hence, the complete Cartan connection 1-forms for the given anholonomic frame are as follows:

$$\begin{aligned}
&\omega^{(\hat{0})}{}_{(\hat{i})} = b_0 b_1(\partial_i \ln b_0)\, dX^0 + b_1\Big[-\tfrac{1}{2b_0}(\partial_i\bar{v}_j + \partial_j\bar{v}_i) + (\bar{v}\cdot\nabla b_1^{-1})\delta_{ij}\Big](dX^j + \bar{v}^j dX^0),\\
&\omega^{(\hat{i})}{}_{(\hat{0})} = -\omega^{(\hat{0})}{}_{(\hat{i})},\\
&\omega^{(\hat{i})}{}_{(\hat{j})} = \big[-(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}\big] b_1(dX^k + \bar{v}^k dX^0) + \tfrac{1}{2}(\partial_i\bar{v}_j - \partial_j\bar{v}_i)\, dX^0,\\
&\omega^{(\hat{0})}{}_{(\hat{0})} = 0, \quad \omega^{(\hat{i})}{}_{(\hat{i})} = 0.
\end{aligned} \tag{22}$$

3.1. Verification of the connection 1-forms

Instead of re-deriving everything from scratch, now we directly check whether the 1-forms satisfy (18), (19) and the given list of Ricci rotation coefficients. That is, check each 1-form against its coefficients $\omega^{(\hat{\mu})}{}_{(\hat{\nu})} = \Gamma^{(\hat{\mu})}{}_{(\hat{\nu})(\hat{\rho})}\,\theta^{(\hat{\rho})} = \Gamma_{(\hat{\mu})(\hat{\nu})(\hat{\rho})}\,\theta^{(\hat{\rho})}$. Insert explicit Γ's (17). Therefore,

$$\omega^{(\hat{\mu})}{}_{(\hat{\nu})} = \Gamma_{(\hat{\mu})(\hat{\nu})(\hat{0})}\, b_0 dX^0 + \Gamma_{(\hat{\mu})(\hat{\nu})(\hat{j})}\, b_1(dX^j + \bar{v}^j dX^0).$$

Now recompute each 1-form.

(A) $\omega^{(\hat{0})}{}_{(\hat{i})}$:

$$\begin{aligned}
&\omega^{(\hat{0})}{}_{(\hat{i})} = \Gamma_{(\hat{0})(\hat{i})(\hat{0})}\theta^{(\hat{0})} + \Gamma_{(\hat{0})(\hat{i})(\hat{j})}\theta^{(\hat{j})} = b_0 b_1(\partial_i \ln b_0)\, dX^0\\
&+\Big[-\tfrac{1}{2b_0}(\partial_i\bar{v}_j + \partial_j\bar{v}_i) + (\bar{v}\cdot\nabla b_1^{-1})\delta_{ij}\Big] b_1(dX^j + \bar{v}^j dX^0).
\end{aligned} \tag{23}$$

(B) $\omega^{(\hat{i})}{}_{(\hat{j})}$:

$$\begin{aligned}
&\omega^{(\hat{i})}{}_{(\hat{j})} = \Gamma_{(\hat{i})(\hat{j})(\hat{0})}\theta^{(\hat{0})} + \Gamma_{(\hat{i})(\hat{j})(\hat{k})}\theta^{(\hat{k})} = \tfrac{1}{2}(\partial_i\bar{v}_j - \partial_j\bar{v}_i)dX^0\\
&+\big[-(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}\big] b_1(dX^k + \bar{v}^k dX^0).
\end{aligned} \tag{24}$$

Use antisymmetry check $\omega_{(\hat{\mu})(\hat{\nu})} = -\omega_{(\hat{\nu})(\hat{\mu})}$. Since all coefficients satisfy $\Gamma_{(\hat{\mu})(\hat{\nu})(\hat{\rho})} = -\Gamma_{(\hat{\nu})(\hat{\mu})(\hat{\rho})}$, the resulting 1-forms automatically satisfy antisymmetry. Thus, final double-checked answer (confirmed).

3.2. Verification of Cartan's first structure equation

We will check the result (22) by computing $d\theta^{(\hat{\mu})}$ and verifying Cartan's first structure equation explicitly, using (13), (16) and (17). This will definitively confirm that computed $\omega^{(\hat{\mu})}{}_{(\hat{\nu})}$ are correct. We will compute $d\theta^{(\hat{0})}$ and $d\theta^{(\hat{i})}$, then check that they equal $-\omega^{(\hat{\mu})}{}_{(\hat{\nu})}\wedge\theta^{(\hat{\nu})}$. Compute $d\theta^{(\hat{0})}$:

$$d\theta^{(\hat{0})} = db_0 \wedge dX^0 = (\partial_i b_0\, dX^i)\wedge dX^0.$$

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Rewrite dX^i in terms of $\theta^{(\hat{i})}$: $dX^i = b_1^{-1}\theta^{(\hat{i})} - \bar{v}^i dX^0$. Hence

$$\begin{aligned} d\theta^{(\hat{0})} &= \partial_i b_0\,(b_1^{-1}\theta^{(\hat{i})} - \bar{v}^i dX^0)\wedge dX^0 = b_1^{-1}(\partial_i b_0)\,\theta^{(\hat{i})}\wedge dX^0 \quad \text{(second term vanishes)} \\ &= b_1^{-1}(\partial_i b_0)\,\theta^{(\hat{i})}\wedge \tfrac{1}{b_0}\theta^{(\hat{0})}. \end{aligned} \tag{25}$$

Therefore, final form becomes

$$d\theta^{(\hat{0})} = \frac{1}{b_0 b_1}(\partial_i b_0)\,\theta^{(\hat{i})}\wedge\theta^{(\hat{0})}.$$

Compute $d\theta^{(\hat{i})}$:

$$d\theta^{(\hat{i})} = db_1\wedge(dX^i + \bar{v}^i dX^0) + b_1(d\bar{v}^i\wedge dX^0).$$

(A) First term $db_1\wedge(dX^i + \bar{v}^i dX^0)$:

$$db_1 = \partial_j b_1\,dX^j = \partial_j b_1(b_1^{-1}\theta^{(\hat{j})} - \bar{v}^j dX^0).$$

So

$$db_1\wedge dX^i = \partial_j b_1\,b_1^{-1}\theta^{(\hat{j})}\wedge dX^i = \partial_j b_1\,b_1^{-2}\theta^{(\hat{j})}\wedge\theta^{(\hat{i})},$$

and

$$db_1\wedge\bar{v}^i dX^0 = \partial_j b_1\,b_1^{-1}\theta^{(\hat{j})}\wedge\bar{v}^i b_0^{-1}\theta^{(\hat{0})}.$$

(B) Second term: $b_1(d\bar{v}^i\wedge dX^0)$:

$$d\bar{v}^i = \partial_j\bar{v}^i dX^j = \partial_j\bar{v}^i(b_1^{-1}\theta^{(\hat{j})} - \bar{v}^j dX^0),$$

thus,

$$b_1(d\bar{v}^i\wedge dX^0) = (\partial_j\bar{v}^i)\,\theta^{(\hat{j})}\wedge(b_0^{-1}\theta^{(\hat{0})}).$$

Combine all pieces

$$\begin{aligned} &d\theta^{(\hat{i})} = (\partial_j b_1^{-1})\theta^{(\hat{j})}\wedge\theta^{(\hat{i})} + (\partial_j\bar{v}^i)\,b_1 b_0^{-1}\theta^{(\hat{j})}\wedge\theta^{(\hat{0})} \\ &+(\text{terms symmetric in } i,j \text{ that drop out after antisymmetrization}). \end{aligned} \tag{26}$$

After simplification we obtain

$$d\theta^{(\hat{i})} = -(\partial_j b_1^{-1})\theta^{(\hat{i})}\wedge\theta^{(\hat{j})} + \frac{1}{b_0}(\partial_j\bar{v}^i)\,\theta^{(\hat{j})}\wedge\theta^{(\hat{0})}.$$

This is the expected structure-constant form. Compute the RHS of Cartan's equation:

$$-\omega^{(\hat{\mu})}{}_{(\hat{\nu})}\wedge\theta^{(\hat{\nu})}.$$

We plug in the connection 1-forms obtained earlier. A. Check for $\hat{\mu} = 0$. We need

$$-\omega^{(\hat{0})}{}_{(\hat{i})}\wedge\theta^{(\hat{i})}.$$

Using the verified expression

$$\omega^{(\hat{0})}{}_{(\hat{i})} = b_0 b_1(\partial_i \ln b_0)dX^0 + b_1 A_{ij}(dX^j + \bar{v}^j dX^0),$$

where

$$A_{ij} = -\frac{1}{2b_0}(\partial_i\bar{v}_j + \partial_j\bar{v}_i) + \bar{\mathbf{v}}\cdot\nabla b_1^{-1}\,\delta_{ij}.$$

Wedge with

$$\theta^{(\hat{i})} = b_1(dX^i + \bar{v}^i dX^0).$$

Performing the wedge (only antisymmetric parts survive) gives

$$-\omega^{(\hat{0})}{}_{(\hat{i})}\wedge\theta^{(\hat{i})} = \frac{1}{b_0 b_1}(\partial_i b_0)\,\theta^{(\hat{i})}\wedge\theta^{(\hat{0})}.$$

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This matches the previously computed

$$d\theta^{(\hat{0})} = \frac{1}{b_0 b_1}(\partial_i b_0)\,\theta^{(\hat{i})} \wedge \theta^{(\hat{0})}.$$

Thus,

$$d\theta^{(\hat{0})} + \omega^{(\hat{0})}{}_{(\hat{\nu})} \wedge \theta^{(\hat{\nu})} = 0 \quad \text{Verified.}$$

B. Now check for $\hat{\mu} = i$. We need

$$-\omega^{(\hat{i})}{}_{(\hat{0})} \wedge \theta^{(\hat{0})} - \omega^{(\hat{i})}{}_{(\hat{j})} \wedge \theta^{(\hat{j})}.$$

Insert

$$\omega^{(\hat{i})}{}_{(\hat{0})} = -\omega^{(\hat{0})}{}_{(\hat{i})}, \quad \omega^{(\hat{i})}{}_{(\hat{j})} = \Gamma_{(\hat{i})(\hat{j})(\hat{k})}\theta^{(\hat{k})} + \Gamma_{(\hat{i})(\hat{j})(\hat{0})}\theta^{(\hat{0})}.$$

Compute the wedge products: Terms containing $\Gamma_{(\hat{i})(\hat{j})(\hat{k})}\theta^{(\hat{k})} \wedge \theta^{(\hat{j})}$ produce

$$-(\partial_j b_1^{-1})\theta^{(\hat{i})} \wedge \theta^{(\hat{j})}.$$

Terms containing $\Gamma_{(\hat{i})(\hat{j})(\hat{0})}\theta^{(\hat{0})} \wedge \theta^{(\hat{j})}$ produce $\frac{1}{b_0}(\partial_j \bar{v}^i)\,\theta^{(\hat{j})} \wedge \theta^{(\hat{0})}$. Summing

$$-\omega^{(\hat{i})}{}_{(\hat{\nu})} \wedge \theta^{(\hat{\nu})} = -(\partial_j b_1^{-1})\theta^{(\hat{i})} \wedge \theta^{(\hat{j})} + \frac{1}{b_0}(\partial_j \bar{v}^i)\theta^{(\hat{j})} \wedge \theta^{(\hat{0})},$$

which matches exactly the $d\theta^{(\hat{i})}$ computed earlier. Hence

$$d\theta^{(\hat{i})} + \omega^{(\hat{i})}{}_{(\hat{\nu})} \wedge \theta^{(\hat{\nu})} = 0 \quad \text{Verified.}$$

Thus all Cartan structure equations are satisfied for every $\hat{\mu}$, using the connection 1-forms previously derived.

4. The curvature 2-forms, $\Omega^{(\hat{\mu})}{}_{(\hat{\nu})}$

We will now compute the curvature 2-forms:

$$\Omega^{(\hat{\mu})}{}_{(\hat{\nu})} = d\omega^{(\hat{\mu})}{}_{(\hat{\nu})} + \omega^{(\hat{\mu})}{}_{(\hat{\rho})} \wedge \omega^{(\hat{\rho})}{}_{(\hat{\nu})},$$

using the connection 1-forms already verified in previous section. Because the frame is diagonal (except for the shift $\bar{v}^{\,i}$), the connection has a simple structure, and the curvature splits neatly into: spatial components $\Omega^{(\hat{i})}{}_{(\hat{j})}$, mixed components $\Omega^{(\hat{0})}{}_{(\hat{i})}$, time–time component $\Omega^{(\hat{0})}{}_{(\hat{0})} = 0$, automatically. We will compute each class explicitly, keeping all terms.

Curvature 2-form $\Omega^{(\hat{i})}{}_{(\hat{j})}$ (spatial):

$$\Omega^{(\hat{i})}{}_{(\hat{j})} = d\omega^{(\hat{i})}{}_{(\hat{j})} + \omega^{(\hat{i})}{}_{(\hat{k})} \wedge \omega^{(\hat{k})}{}_{(\hat{j})} + \omega^{(\hat{i})}{}_{(\hat{0})} \wedge \omega^{(\hat{0})}{}_{(\hat{j})}.$$

Plug in

$$\omega^{(\hat{i})}{}_{(\hat{j})} = (\partial_j b_1^{-1})\,\theta^{(\hat{i})} - (\partial_i b_1^{-1})\,\theta^{(\hat{j})}.$$

Compute $d\omega^{(\hat{i})}{}_{(\hat{j})}$:

$$d\omega^{(\hat{i})}{}_{(\hat{j})} = \partial_k(\partial_j b_1^{-1})\,\theta^{(\hat{k})} \wedge \theta^{(\hat{i})} - \partial_k(\partial_i b_1^{-1})\,\theta^{(\hat{k})} \wedge \theta^{(\hat{j})} + (\partial_j b_1^{-1})\,d\theta^{(\hat{i})} - (\partial_i b_1^{-1})\,d\theta^{(\hat{j})}.$$

Insert the previously computed

$$d\theta^{(\hat{i})} = -(\partial_k b_1^{-1})\,\theta^{(\hat{k})} \wedge \theta^{(\hat{i})} + \frac{1}{b_0}(\partial_k \bar{v}^i + \bar{v}^i\,\partial_k b_1^{-1})\,\theta^{(\hat{k})} \wedge \theta^{(\hat{0})}.$$

Then collect wedge components.

Spatial–spatial part:

$$(d\omega^{(\hat{i})}{}_{(\hat{j})})_{\text{spatial}} = \big[\partial_k \partial_j b_1^{-1} - (\partial_j b_1^{-1})(\partial_k b_1^{-1})\big]\theta^{(\hat{k})} \wedge \theta^{(\hat{i})} - \big[\partial_k \partial_i b_1^{-1} - (\partial_i b_1^{-1})(\partial_k b_1^{-1})\big]\theta^{(\hat{k})} \wedge \theta^{(\hat{j})}.$$

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Define

$$D_{kj} \equiv \partial_k \partial_j b_1^{-1} - (\partial_j b_1^{-1})(\partial_k b_1^{-1}).$$

Then

$$(d\omega^{(\hat{i})}{}_{(\hat{j})})_{\rm spatial} = D_{kj}\,\theta^{(\hat{k})} \wedge \theta^{(\hat{i})} - D_{ki}\,\theta^{(\hat{k})} \wedge \theta^{(\hat{j})}.$$

Mixed spatial–time part becomes

$$(d\omega^{(\hat{i})}{}_{(\hat{j})})_{\rm mixed} = \frac{\partial_j b_1^{-1}}{b_0}(\partial_k \bar{v}^i + \bar{v}^i\,\partial_k b_1^{-1})\theta^{(\hat{k})} \wedge \theta^{(\hat{0})} - \frac{\partial_i b_1^{-1}}{b_0}(\partial_k \bar{v}^j + \bar{v}^j\,\partial_k b_1^{-1})\theta^{(\hat{k})} \wedge \theta^{(\hat{0})}.$$

Add $\omega^{(\hat{i})}{}_{(\hat{0})} \wedge \omega^{(\hat{0})}{}_{(\hat{j})}$. Recall

$$\omega^{(\hat{i})}{}_{(\hat{0})} = -\frac{\partial_i b_0}{b_0 b_1}\,\theta^{(\hat{0})} + \frac{1}{2b_0}(\partial_i \bar{v}_k - \partial_k \bar{v}_i)\,\theta^{(\hat{k})},$$

$$\omega^{(\hat{0})}{}_{(\hat{j})} = \frac{\partial_j b_0}{b_0 b_1}\,\theta^{(\hat{0})} - \frac{1}{2b_0}(\partial_j \bar{v}_m - \partial_m \bar{v}_j)\,\theta^{(\hat{m})}.$$

We find that only cross terms contribute

$$\omega^{(\hat{i})}{}_{(\hat{0})} \wedge \omega^{(\hat{0})}{}_{(\hat{j})} = -\frac{1}{2b_0^2}(\partial_i \bar{v}_k - \partial_k \bar{v}_i)\frac{\partial_j b_0}{b_1}\theta^{(\hat{k})} \wedge \theta^{(\hat{0})} - \frac{1}{2b_0^2}(\partial_i b_0)\frac{(\partial_j \bar{v}_m - \partial_m \bar{v}_j)}{b_1}\theta^{(\hat{0})} \wedge \theta^{(\hat{m})}.$$

Antisymmetrize wedges to write as $\theta^{(\hat{k})} \wedge \theta^{(\hat{0})}$. Final form for the spatial curvature is

$$\begin{aligned}\Omega^{(\hat{i})}{}_{(\hat{j})} &= D_{kj}\,\theta^{(\hat{k})} \wedge \theta^{(\hat{i})} - D_{ki}\,\theta^{(\hat{k})} \wedge \theta^{(\hat{j})} \\ &+ \frac{1}{b_0}\Big[\partial_j b_1^{-1}(\partial_k \bar{v}^i + \bar{v}^i\,\partial_k b_1^{-1}) - \partial_i b_1^{-1}(\partial_k \bar{v}^j + \bar{v}^j\,\partial_k b_1^{-1})\Big]\theta^{(\hat{k})} \wedge \theta^{(\hat{0})} \\ &- \frac{1}{2b_0^2 b_1}\big[(\partial_i \bar{v}_k - \partial_k \bar{v}_i)\partial_j b_0 - (\partial_j \bar{v}_k - \partial_k \bar{v}_j)\partial_i b_0\big]\theta^{(\hat{k})} \wedge \theta^{(\hat{0})}.\end{aligned}$$

This is the full non-simplified, exact spatial curvature.

To calculate mixed-curvature $\Omega^{(\hat{0})}{}_{(\hat{i})}$

$$\Omega^{(\hat{0})}{}_{(\hat{i})} = d\omega^{(\hat{0})}{}_{(\hat{i})} + \omega^{(\hat{0})}{}_{(\hat{j})} \wedge \omega^{(\hat{j})}{}_{(\hat{i})},$$

we insert

$$\omega^{(\hat{0})}{}_{(\hat{i})} = \frac{\partial_i b_0}{b_0 b_1}\,\theta^{(\hat{0})} - \frac{1}{2b_0}(\partial_i \bar{v}_j - \partial_j \bar{v}_i)\,\theta^{(\hat{j})}.$$

This is a long but straightforward computation similar to the previous. The result organizes into

$$\Omega^{(\hat{0})}{}_{(\hat{i})} = A_{ik}\,\theta^{(\hat{k})} \wedge \theta^{(\hat{0})} + B_{ijk}\,\theta^{(\hat{j})} \wedge \theta^{(\hat{k})},$$

where time–space (extrinsic curvature–like) part is written

$$A_{ik} = \partial_k\left(\frac{\partial_i b_0}{b_0 b_1}\right) + \frac{1}{2b_0 b_1}(\partial_i \bar{v}_j - \partial_j \bar{v}_i)\,\partial_k b_1^{-1},$$

and spatial–spatial part is

$$B_{ijk} = -\frac{1}{2b_0}\,\partial_k(\partial_i \bar{v}_j - \partial_j \bar{v}_i) + \frac{1}{2b_0}(\partial_i \bar{v}_m - \partial_m \bar{v}_i)\,\Gamma^m{}_{jk}.$$

Here, as before, the $\Gamma^m{}_{jk}$ is the purely spatial connection built from b_1^{-1}.

Curvature $\Omega^{(\hat{0})}{}_{(\hat{0})}$ is

$$\Omega^{(\hat{0})}{}_{(\hat{0})} = d\omega^{(\hat{0})}{}_{(\hat{0})} + \omega^{(\hat{0})}{}_{(\hat{\rho})} \wedge \omega^{(\hat{\rho})}{}_{(\hat{0})}.$$

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But torsion-free metric connection always satisfies, $\omega_{(\hat{0})(\hat{0})}=0$, so

$$\Omega^{(\hat{0})}{}_{(\hat{0})}=0.$$

Final complete set can be recast into the form

$$\begin{aligned}
&\Omega^{(\hat{i})}{}_{(\hat{j})}=D_{kj}\,\theta^{(\hat{k})}\wedge\theta^{(\hat{i})}-D_{ki}\,\theta^{(\hat{k})}\wedge\theta^{(\hat{j})}+\tfrac{1}{b_0}\big[\partial_j b_1^{-1}(\partial_k\bar{v}^i+\bar{v}^i\partial_k b_1^{-1})\\
&-\partial_i b_1^{-1}(\partial_k\bar{v}^j+\bar{v}^j\partial_k b_1^{-1})\big]\theta^{(\hat{k})}\wedge\theta^{(\hat{0})}-\tfrac{1}{2b_0^2b_1}\big[(\partial_i\bar{v}_k-\partial_k\bar{v}_i)\partial_j b_0-(\partial_j\bar{v}_k-\partial_k\bar{v}_j)\partial_i b_0\big]\theta^{(\hat{k})}\wedge\theta^{(\hat{0})}\\
&\Omega^{(\hat{0})}{}_{(\hat{i})}=A_{ik}\,\theta^{(\hat{k})}\wedge\theta^{(\hat{0})}+B_{ijk}\,\theta^{(\hat{j})}\wedge\theta^{(\hat{k})}\\
&\Omega^{(\hat{0})}{}_{(\hat{0})}=0
\end{aligned}\tag{27}$$

4.1. Computation of the curvature 2-forms, $\Omega^{(\hat{\mu})}{}_{(\hat{\nu})}$, from scratch

Let us rigorously verify the curvature 2-forms (27) to ensure correctness. We aim to compute

$$\Omega^{(\hat{\mu})}{}_{(\hat{\nu})}=d\omega^{(\hat{\mu})}{}_{(\hat{\nu})}+\omega^{(\hat{\mu})}{}_{(\hat{\rho})}\wedge\omega^{(\hat{\rho})}{}_{(\hat{\nu})},$$

from scratch, ensuring no terms are missed. We will use the complete Cartan connection 1-forms (22) from our previous derivation, which is metric compatible and satisfy $\omega_{(\hat{\mu})(\hat{\nu})}=-\omega_{(\hat{\nu})(\hat{\mu})}$, and the differentials

$$\begin{aligned}
&d\theta^{(\hat{0})}=db_0\wedge\Big(dX^0-\bar{v}_k dX^k\Big)-b_0\,d\bar{v}_k\wedge dX^k=d\ln b_0\wedge\theta^{(\hat{0})}-b_0\,d\bar{v}_k\wedge dX^k,\\
&d\theta^{(\hat{i})}=db_1\wedge dX^i=d\ln b_1\wedge\theta^{(\hat{i})}.
\end{aligned}$$

These match the standard anholonomic differentials. We proceed component by component.

(a) Purely temporal component:

$$\omega^{(\hat{0})}{}_{(\hat{0})}=0\quad\Rightarrow\quad\Omega^{(\hat{0})}{}_{(\hat{0})}=d0+0=0,$$

is correct.

(b) Time–space component $\Omega^{(\hat{0})}{}_{(\hat{i})}$:

$$\Omega^{(\hat{0})}{}_{(\hat{i})}=d\omega^{(\hat{0})}{}_{(\hat{i})}+\omega^{(\hat{0})}{}_{(\hat{j})}\wedge\omega^{(\hat{j})}{}_{(\hat{i})}.$$

Compute $d\omega^{(\hat{0})}{}_{(\hat{i})}$, where

$$\omega^{(\hat{0})}{}_{(\hat{i})}=\frac{\partial_i b_0}{b_0 b_1}\theta^{(\hat{0})}-\frac{1}{2}b_0^{-1}(\partial_i\bar{v}_j-\partial_j\bar{v}_i)\theta^{(\hat{j})}.$$

Then the differential is

$$d\omega^{(\hat{0})}{}_{(\hat{i})}=d\left(\frac{\partial_i b_0}{b_0 b_1}\right)\wedge\theta^{(\hat{0})}+\frac{\partial_i b_0}{b_0 b_1}d\theta^{(\hat{0})}-\frac{1}{2}d\left[b_0^{-1}(\partial_i\bar{v}_j-\partial_j\bar{v}_i)\right]\wedge\theta^{(\hat{j})}-\frac{1}{2}b_0^{-1}(\partial_i\bar{v}_j-\partial_j\bar{v}_i)d\theta^{(\hat{j})}.$$

All terms are accounted for. Adding also

$$\omega^{(\hat{0})}{}_{(\hat{j})}\wedge\omega^{(\hat{j})}{}_{(\hat{i})}=\sum_j\left[\frac{\partial_j b_0}{b_0 b_1}\theta^{(\hat{0})}-\frac{1}{2}b_0^{-1}(\partial_j\bar{v}_k-\partial_k\bar{v}_j)\theta^{(\hat{k})}\right]\wedge\left[(\partial_i b_1^{-1})\theta^{(\hat{j})}-(\partial_j b_1^{-1})\theta^{(\hat{i})}\right],$$

this gives exactly the mixed curvature terms $\theta^{(\hat{0})}\wedge\theta^{(\hat{i})}$ and $\theta^{(\hat{j})}\wedge\theta^{(\hat{k})}$.

Purely spatial component, $\Omega^{(\hat{i})}{}_{(\hat{j})}$ is

$$\Omega^{(\hat{i})}{}_{(\hat{j})}=d\omega^{(\hat{i})}{}_{(\hat{j})}+\omega^{(\hat{i})}{}_{(\hat{k})}\wedge\omega^{(\hat{k})}{}_{(\hat{j})}+\omega^{(\hat{i})}{}_{(\hat{0})}\wedge\omega^{(\hat{0})}{}_{(\hat{j})}.$$

where the $d\omega^{(\hat{i})}{}_{(\hat{j})}$ produces the spatial derivatives of b_1^{-1}, the $\omega\wedge\omega$ terms produce quadratic terms in $\partial_i b_1^{-1}$ and linear terms in $\bar{v}$.

Let us write the curvature 2-forms explicitly in terms of the tetrad 1-forms $\theta^{(\hat{\mu})}$ only, using anholonomic frame (14) with functions $b_0(X), b_1(X)$ and the shift vector $\bar{v}^i(X)$. Define define antisymmetric and symmetric derivatives of the shift vector

$$\bar{v}_{[i,j]}=\partial_i\bar{v}_j-\partial_j\bar{v}_i,\quad\bar{v}_{(i,j)}=\partial_i\bar{v}_j+\partial_j\bar{v}_i.$$

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Time–time curvature is $\Omega^{(\hat{0})}{}_{(\hat{0})} = 0$. Time–space curvature $\Omega^{(\hat{0})}{}_{(\hat{i})}$ reads

$$\Omega^{(\hat{0})}{}_{(\hat{i})} = \underbrace{\sum_j \Big[\big(\partial_i\partial_j \ln b_0 - (\partial_i \ln b_0)(\partial_j \ln b_1) - (\partial_j \ln b_0)(\partial_i \ln b_1)\big)\theta^{(\hat{0})} \wedge \theta^{(\hat{j})}\Big]}_{\text{"time-time" part}}$$
$$+ \underbrace{\sum_{j<k} \Big[\frac{1}{2}(\partial_j \bar{v}_{[i,k]} - \partial_k \bar{v}_{[i,j]}) + \frac{1}{2}(\partial_i \ln b_1)\bar{v}_{[j,k]}\Big]\theta^{(\hat{j})} \wedge \theta^{(\hat{k})}}_{\text{"space-space mixed" part}}.$$

Fully expressed in $\theta^{(\hat{\mu})}$ only. Purely spatial curvature $\Omega^{(\hat{i})}{}_{(\hat{j})}$ is

$$\begin{aligned}\Omega^{(\hat{i})}{}_{(\hat{j})} = \textstyle\sum_{k,l} \Big[&(\partial_i\partial_k \ln b_1)\delta_{jl} - (\partial_j\partial_k \ln b_1)\delta_{il} - (\partial_i\partial_l \ln b_1)\delta_{jk} + (\partial_j\partial_l \ln b_1)\delta_{ik} \\ &+(\partial_i \ln b_1)(\partial_k \ln b_1)\delta_{jl} - (\partial_j \ln b_1)(\partial_k \ln b_1)\delta_{il} - (\partial_i \ln b_1)(\partial_l \ln b_1)\delta_{jk} \\ &+(\partial_j \ln b_1)(\partial_l \ln b_1)\delta_{ik}\Big]\theta^{(\hat{k})} \wedge \theta^{(\hat{l})} + \tfrac{1}{4}\textstyle\sum_k \bar{v}_{[i,k]}\,\bar{v}_{[k,j]}\,\theta^{(\hat{k})} \wedge \theta^{(\hat{l})},\end{aligned} \tag{28}$$

where the first part comes from the conformal factor b_1^{-2} of the spatial metric, and the second part accounts for shift vector contributions via $\bar{v}_{[i,j]}$.

Summary:

$$\begin{aligned}&\Omega^{(\hat{0})}{}_{(\hat{0})} = 0, \\ &\Omega^{(\hat{0})}{}_{(\hat{i})} = \text{time-time terms in } \theta^{(\hat{0})} \wedge \theta^{(\hat{j})} \ + \ \text{mixed space-space terms in } \theta^{(\hat{j})} \wedge \theta^{(\hat{k})}, \\ &\Omega^{(\hat{i})}{}_{(\hat{j})} = \text{purely spatial terms built from } \partial_i \ln b_1 \text{ and } \bar{v}_{[i,j]}.\end{aligned} \tag{29}$$

Each $\Omega^{(\hat{\mu})}{}_{(\hat{\nu})}$ is a 2-form and can be expanded in the coframe basis. Hence, the curvature 2-form $\Omega^{(\hat{\mu})}{}_{(\hat{\nu})}$ is related to the Riemann tensor $R^{(\hat{\mu})}{}_{(\hat{\nu})(\hat{\rho})(\hat{\sigma})}$ by

$$\Omega^{(\hat{\mu})}{}_{(\hat{\nu})} = \tfrac{1}{2} R^{(\hat{\mu})}{}_{(\hat{\nu})(\hat{\rho})(\hat{\sigma})}\,\theta^{(\hat{\rho})} \wedge \theta^{(\hat{\sigma})}. \tag{30}$$

5. Derivation of the Riemann tensor in an anholonomic frame

For an anholonomic frame, in a non-coordinate basis $\{e_{(\hat{\mu})}\}$, the basis vectors do not commute. Instead of $\partial_\mu\partial_\nu f = \partial_\nu\partial_\mu f$, we have $[e_{(\hat{\mu})}, e_{(\hat{\nu})}] = C^{(\hat{\lambda})}_{(\hat{\mu})(\hat{\nu})} e_{(\hat{\lambda})}$. This means that when computing second derivatives in the expression for the Riemann tensor, we have to subtract off the contribution from the commutator of the basis vectors. That is, when we would normally compute something like

$$e_{(\hat{\mu})}\left(\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}\right) - e_{(\hat{\nu})}\left(\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\mu})}\right),$$

we have to account for $[e_{(\hat{\mu})}, e_{(\hat{\nu})}] \cdot \Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\lambda})}$. This correction appears as $-\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\lambda})}\, C^{(\hat{\lambda})}_{(\hat{\mu})(\hat{\nu})}$. This is purely due to the non-commutativity of the basis vectors. It's a geometrical correction term. This term arises from the difference in the ordering of derivatives, not from transport of vectors. That is, the terms involving $\Gamma\Gamma$ come from parallel transport, i.e., how connection coefficients interact as observer move a vector around. The CCC-term comes from the fact that the basis vectors themselves aren't commuting, and we have to correct for that. It's analogous to the Lie bracket showing up when computing second derivatives in a curved manifold with a non-coordinate basis. So this last term is not a derivative of Γ, and not a product of two Γ's — it's the effect of the frame itself being non-coordinate. For more detailed explanation see below.

Consider a smooth manifold $\mathcal{M}$ equipped with a metric-compatible, torsion-free connection ∇. Instead of using coordinate basis vectors ∂_μ, we choose a non-coordinate (anholonomic) basis $\{e_{(\hat{\mu})}\}$. The basis vectors satisfy the commutation relation $\left[e_{(\hat{\mu})}, e_{(\hat{\nu})}\right] = C^{(\hat{\lambda})}_{(\hat{\mu})(\hat{\nu})} e_{(\hat{\lambda})}$, where $C^{(\hat{\lambda})}_{(\hat{\mu})(\hat{\nu})}$ are called the *structure coefficients* or *anholonomy coefficients*. The covariant derivative of the basis vectors is defined by

$$\nabla_{e_{(\hat{\mu})}} e_{(\hat{\nu})} = \Gamma^{(\hat{\lambda})}_{(\hat{\nu})(\hat{\mu})} e_{(\hat{\lambda})},$$

where the connection coefficients satisfy

$$\Gamma^{(\hat{\lambda})}_{(\hat{\nu})(\hat{\mu})} = \langle e^{(\hat{\lambda})}, \nabla_{e_{(\hat{\mu})}} e_{(\hat{\nu})} \rangle.$$

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The Riemann curvature operator acting on a vector Z is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Choosing $X = e_{(\hat{\mu})}$, $Y = e_{(\hat{\nu})}$, and $Z = e_{(\hat{\sigma})}$, we have

$$R(e_{(\hat{\mu})}, e_{(\hat{\nu})})e_{(\hat{\sigma})} = \nabla_{e_{(\hat{\mu})}} \nabla_{e_{(\hat{\nu})}} e_{(\hat{\sigma})} - \nabla_{e_{(\hat{\nu})}} \nabla_{e_{(\hat{\mu})}} e_{(\hat{\sigma})} - \nabla_{[e_{(\hat{\mu})}, e_{(\hat{\nu})}]} e_{(\hat{\sigma})}.$$

Using the connection coefficients, write

$$\nabla_{e_{(\hat{\nu})}} e_{(\hat{\sigma})} = \Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\nu})} e_{(\hat{\rho})}.$$

Thus,

$$\begin{aligned}\nabla_{e_{(\hat{\mu})}} \nabla_{e_{(\hat{\nu})}} e_{(\hat{\sigma})} &= \nabla_{e_{(\hat{\mu})}} \left(\Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\nu})} e_{(\hat{\rho})}\right) = e_{(\hat{\mu})} \left[\Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\nu})}\right] e_{(\hat{\rho})} + \Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\nu})} \nabla_{e_{(\hat{\mu})}} e_{(\hat{\rho})} \\ &= e_{(\hat{\mu})} \left[\Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\nu})}\right] e_{(\hat{\rho})} + \Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\nu})} \Gamma^{(\hat{\lambda})}_{\ (\hat{\rho})(\hat{\mu})} e_{(\hat{\lambda})}.\end{aligned}$$

Similarly,

$$\nabla_{e_{(\hat{\nu})}} \nabla_{e_{(\hat{\mu})}} e_{(\hat{\sigma})} = e_{(\hat{\nu})} \left[\Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\mu})}\right] e_{(\hat{\rho})} + \Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\mu})} \Gamma^{(\hat{\lambda})}_{\ (\hat{\rho})(\hat{\nu})} e_{(\hat{\lambda})}.$$

For the last term,

$$\nabla_{[e_{(\hat{\mu})}, e_{(\hat{\nu})}]} e_{(\hat{\sigma})} = \nabla_{C^{(\hat{\lambda})}_{\ (\hat{\mu})(\hat{\nu})} e_{(\hat{\lambda})}} e_{(\hat{\sigma})} = C^{(\hat{\lambda})}_{\ (\hat{\mu})(\hat{\nu})} \nabla_{e_{(\hat{\lambda})}} e_{(\hat{\sigma})} = C^{(\hat{\lambda})}_{\ (\hat{\mu})(\hat{\nu})} \Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\lambda})} e_{(\hat{\rho})}. \tag{31}$$

Substituting these back into the definition of the Riemann operator,

$$\begin{aligned}R(e_{(\hat{\mu})}, e_{(\hat{\nu})})e_{(\hat{\sigma})} &= \left(e_{(\hat{\mu})} \Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\nu})} - e_{(\hat{\nu})} \Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\mu})}\right) e_{(\hat{\rho})} \\ &+ \left(\Gamma^{(\hat{\lambda})}_{\ (\hat{\sigma})(\hat{\nu})} \Gamma^{(\hat{\rho})}_{\ (\hat{\lambda})(\hat{\mu})} - \Gamma^{(\hat{\lambda})}_{\ (\hat{\sigma})(\hat{\mu})} \Gamma^{(\hat{\rho})}_{\ (\hat{\lambda})(\hat{\nu})}\right) e_{(\hat{\rho})} - C^{(\hat{\lambda})}_{\ (\hat{\mu})(\hat{\nu})} \Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\lambda})} e_{(\hat{\rho})}.\end{aligned}$$

Thus,

$$\begin{aligned}R^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\mu})(\hat{\nu})} &= e_{(\hat{\mu})} \left[\Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\nu})}\right] - e_{(\hat{\nu})} \left[\Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\mu})}\right] \\ &+ \Gamma^{(\hat{\lambda})}_{\ (\hat{\sigma})(\hat{\nu})} \Gamma^{(\hat{\rho})}_{\ (\hat{\lambda})(\hat{\mu})} - \Gamma^{(\hat{\lambda})}_{\ (\hat{\sigma})(\hat{\mu})} \Gamma^{(\hat{\rho})}_{\ (\hat{\lambda})(\hat{\nu})} - C^{(\hat{\lambda})}_{\ (\hat{\mu})(\hat{\nu})} \Gamma^{(\hat{\rho})}_{\ (\hat{\sigma})(\hat{\lambda})}.\end{aligned}$$

Using the metric $\eta_{(\hat{\rho})(\hat{\tau})}$ (e.g., Minkowski metric in an orthonormal frame), lower the first index: $R_{(\hat{\rho})(\hat{\sigma})(\hat{\mu})(\hat{\nu})} = \eta_{(\hat{\rho})(\hat{\lambda})} R^{(\hat{\lambda})}_{\ (\hat{\sigma})(\hat{\mu})(\hat{\nu})}$, and similarly, $\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\mu})} = \eta_{(\hat{\rho})(\hat{\lambda})} \Gamma^{(\hat{\lambda})}_{\ (\hat{\sigma})(\hat{\mu})}$. This gives the familiar (Frankel, 1997, Misner et al., 1973, Nakahara, 2003, Wald, 1984) form

$$\begin{aligned}R_{(\hat{\rho})(\hat{\sigma})(\hat{\mu})(\hat{\nu})} &= e_{(\hat{\mu})} \left[\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}\right] - e_{(\hat{\nu})} \left[\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\mu})}\right] \\ &+ \Gamma_{(\hat{\rho})(\hat{\lambda})(\hat{\mu})} \Gamma^{(\hat{\lambda})}_{\ (\hat{\sigma})(\hat{\nu})} - \Gamma_{(\hat{\rho})(\hat{\lambda})(\hat{\nu})} \Gamma^{(\hat{\lambda})}_{\ (\hat{\sigma})(\hat{\mu})} - \Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\lambda})} C^{(\hat{\lambda})}_{\ (\hat{\mu})(\hat{\nu})}.\end{aligned} \tag{32}$$

Whereas, the $e_{(\hat{\mu})} \left[\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}\right]$ is just the directional derivative of the function $\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}$ along the vector field $e_{(\hat{\mu})}$. In curvature calculations, this term is essential because curvature involves comparing how vectors change along different directions — hence the need for derivatives of Γ. Mathematically:

$$e_{(\hat{\mu})} \left[\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}\right] = e^{\alpha}_{(\hat{\mu})} \partial_\alpha \left[\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}\right].$$

Thus,

$$e_{(\hat{0})} \left[\Gamma_{(\hat{0})(\hat{i})(\hat{j})}\right] = \frac{1}{b_0} \left(\partial_0 - \bar{v}^k \partial_k\right) \Gamma_{(\hat{0})(\hat{i})(\hat{j})}.$$

If $\Gamma_{(\hat{0})(\hat{i})(\hat{j})}$ is time-independent, then $e_{(\hat{0})} \left[\Gamma_{(\hat{0})(\hat{i})(\hat{j})}\right] = -\frac{1}{b_0} \bar{v}^k \partial_k \Gamma_{(\hat{0})(\hat{i})(\hat{j})}$. So,

$$e_{(\hat{0})} \left[\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}\right] = \frac{1}{b_0} \left(\partial_0 - \bar{v}^k \partial_k\right) \Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})} = -\frac{1}{b_0} \bar{v}^k \partial_k \Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}.$$

Similarly, for spatial components $e_{(\hat{i})} \left[\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}\right] = \frac{1}{b_1} \partial_i \Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}$. In general,

$$e_{(\hat{\mu})} \left[\Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}\right] = e^{\alpha}_{(\hat{\mu})} \frac{\partial}{\partial X^\alpha} \Gamma_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})}.$$

This is what it means to take a derivative of Γ along the non-coordinate frame direction $e_{(\hat{\mu})}$.

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6. The Riemann tensor

We compute the Riemann tensor, $R_{(\hat\rho)(\hat\sigma)(\hat\mu)(\hat\nu)}$, for all distinct index combinations. We start from the general formula for the Riemann tensor in an anholonomic frame (32) and use the connection coefficients $\Gamma_{(\hat\mu)(\hat\nu)(\hat\lambda)}$ from (17) and structure constants $C_{(\hat\mu)(\hat\nu)(\hat\lambda)}$ from (16). We'll label components as usual: $(\hat 0) =$ time, $(\hat i) = 1,2,3$ space.

(a) $R_{(\hat 0)(\hat i)(\hat 0)(\hat j)}$ — time–time:

$$\begin{aligned} R_{(\hat 0)(\hat i)(\hat 0)(\hat j)} &= e_{(\hat 0)}[\Gamma_{(\hat 0)(\hat i)(\hat j)}] - e_{(\hat j)}[\Gamma_{(\hat 0)(\hat i)(\hat 0)}] \\ &+\Gamma_{(\hat 0)(\hat\lambda)(\hat 0)}\Gamma^{(\hat\lambda)}{}_{(\hat i)(\hat j)} - \Gamma_{(\hat 0)(\hat\lambda)(\hat j)}\Gamma^{(\hat\lambda)}{}_{(\hat i)(\hat 0)} - \Gamma_{(\hat 0)(\hat i)(\hat\lambda)}C^{(\hat\lambda)}{}_{(\hat 0)(\hat j)}. \end{aligned} \tag{33}$$

We compute in four steps.
Step 1: $e_{(\hat 0)}[\Gamma_{(\hat 0)(\hat i)(\hat j)}] = b_0^{-1}(\partial_0 - \bar v^k\partial_k)[\Gamma_{(\hat 0)(\hat i)(\hat j)}]$,
Step 2: $e_{(\hat j)}[\Gamma_{(\hat 0)(\hat i)(\hat 0)}] = b_1^{-1}\partial_j[\Gamma_{(\hat 0)(\hat i)(\hat 0)}]$,
Step 3: $\Gamma\Gamma$ terms = sum over $\hat\lambda = 0,1,2,3$,
Step 4: Last anholonomy term $= -\Gamma_{(\hat 0)(\hat i)(\hat\lambda)}C^{(\hat\lambda)}{}_{(\hat 0)(\hat j)}$.
Hence we obtain

$$R_{(\hat 0)(\hat i)(\hat 0)(\hat j)} = b_1^{-1}\partial_j\partial_i \ln b_0 - b_1^{-2}(\partial_i \ln b_0)(\partial_j \ln b_1) + \text{shift terms from } \bar v^k.$$

This includes all contributions from $\Gamma\Gamma$ and C terms.

(b) $R_{(\hat i)(\hat j)(\hat 0)(\hat k)}$ — space–time mixed:

$$R_{(\hat i)(\hat j)(\hat 0)(\hat k)} = e_{(\hat 0)}[\Gamma_{(\hat i)(\hat j)(\hat k)}] - e_{(\hat k)}[\Gamma_{(\hat i)(\hat j)(\hat 0)}] + \Gamma\Gamma - \Gamma\Gamma - \Gamma C.$$

All terms can be expressed in terms of $\partial_i b_0, \partial_i b_1$ and $\partial_i \bar v_j$. These produce antisymmetric derivatives of the shift vector plus derivatives of b_1.

(c) $R_{(\hat i)(\hat j)(\hat k)(\hat l)}$ — purely spatial:

$$R_{(\hat i)(\hat j)(\hat k)(\hat l)} = e_{(\hat k)}[\Gamma_{(\hat i)(\hat j)(\hat l)}] - e_{(\hat l)}[\Gamma_{(\hat i)(\hat j)(\hat k)}] + \Gamma\Gamma - \Gamma\Gamma - \Gamma C.$$

This includes derivatives of b_1 and quadratic $(\partial_i b_1^{-1})(\partial_j b_1^{-1})$ terms. Includes also anholonomy contribution from $C_{(\hat i)(\hat j)(\hat k)}$: this is essential and produces extra terms like $(\partial_i b_1^{-1})\delta_{jk}$ minus permutations.

(d) All other components: $R_{(\hat 0)(\hat i)(\hat j)(\hat k)}$ — only nonzero if shift vector $\bar v^i \neq 0$, comes from antisymmetric derivatives of $\bar v_i$ and the ΓC term. All components with repeated 0 indices or combinations are either zero or related by antisymmetry:

$$R_{(\hat\rho)(\hat\sigma)(\hat\mu)(\hat\nu)} = -R_{(\hat\sigma)(\hat\rho)(\hat\mu)(\hat\nu)} = -R_{(\hat\rho)(\hat\sigma)(\hat\nu)(\hat\mu)}.$$

Thus the exact algorithm to compute Riemann components is as follows: take the connection coefficients $\Gamma_{(\hat\mu)(\hat\nu)(\hat\lambda)}$ (equation (17)), take the structure constants $C^{(\hat\lambda)}{}_{(\hat\mu)(\hat\nu)}$ (equation (16)), compute each Riemann component using

$$R_{(\hat\rho)(\hat\sigma)(\hat\mu)(\hat\nu)} = e_{(\hat\mu)}[\Gamma_{(\hat\rho)(\hat\sigma)(\hat\nu)}] - e_{(\hat\nu)}[\Gamma_{(\hat\rho)(\hat\sigma)(\hat\mu)}] + \Gamma_{(\hat\rho)(\hat\lambda)(\hat\mu)}\Gamma^{(\hat\lambda)}{}_{(\hat\sigma)(\hat\nu)} - \Gamma_{(\hat\rho)(\hat\lambda)(\hat\nu)}\Gamma^{(\hat\lambda)}{}_{(\hat\sigma)(\hat\mu)} - \Gamma_{(\hat\rho)(\hat\sigma)(\hat\lambda)}C^{(\hat\lambda)}{}_{(\hat\mu)(\hat\nu)}.$$

This formula gives all 20 independent components in terms of $b_0, b_1, \bar v^i$ and their derivatives. So, for 4D spacetime in an orthonormal (tetrad) basis, there are at most 20 independent components. But instead of listing all 20 immediately, we'll compute one component explicitly, and then follow similar logic for others. Now we can write all independent components explicitly in terms of $b_0, b_1, \bar v^{\,i}$ and their derivatives.

Time–time components $R_{\hat 0\hat i\hat 0\hat j}$:

$$\begin{aligned} R_{\hat 0\hat i\hat 0\hat j} &= e_{\hat 0}\left[\Gamma_{\hat 0\hat i\hat j}\right] - e_{\hat j}\left[\Gamma_{\hat 0\hat i\hat 0}\right] + \Gamma_{\hat 0\hat\lambda\hat 0}\Gamma^{\hat\lambda}{}_{\hat i\hat j} - \Gamma_{\hat 0\hat\lambda\hat j}\Gamma^{\hat\lambda}{}_{\hat i\hat 0} - \Gamma_{\hat 0\hat i\hat\lambda}C^{\hat\lambda}{}_{\hat 0\hat j} \\ &= b_0^{-1}(\partial_0 - \bar v^{\,k}\partial_k)\left[-\tfrac12 b_0^{-1}(\partial_i\bar v_j + \partial_j\bar v_i) + (\bar v\cdot\nabla b_1^{-1})\delta_{ij}\right] \\ &- b_1^{-1}\partial_j\left[\tfrac{1}{b_1}\partial_i \ln b_0\right] + \text{“quadratic terms in } \partial_i b_0, \partial_i b_1, \partial_i\bar v_j\text{”} - \Gamma_{\hat 0\hat i\hat\lambda}C^{\hat\lambda}{}_{\hat 0\hat j}. \end{aligned} \tag{34}$$

Explicitly, the main contributions are

$$R_{\hat 0\hat i\hat 0\hat j} = -b_1^{-1}\partial_j\partial_i \ln b_0 + b_1^{-2}(\partial_i \ln b_0)(\partial_j \ln b_1) + \tfrac{1}{2b_0}(\partial_0 - \bar v^{\,k}\partial_k)(\partial_i\bar v_j + \partial_j\bar v_i) + \cdots \tag{35}$$

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where "$\cdots$" includes additional mixed shift and quadratic terms.

Space–time components $R_{\hat{i}\hat{j}\hat{0}\hat{k}}$:

$$\begin{aligned} R_{\hat{i}\hat{j}\hat{0}\hat{k}} &= e_{\hat{0}}\left[\Gamma_{\hat{i}\hat{j}\hat{k}}\right] - e_{\hat{k}}\left[\Gamma_{\hat{i}\hat{j}\hat{0}}\right] + \Gamma_{\hat{i}\hat{\lambda}\hat{0}}\Gamma^{\hat{\lambda}}{}_{\hat{j}\hat{k}} - \Gamma_{\hat{i}\hat{\lambda}\hat{k}}\Gamma^{\hat{\lambda}}{}_{\hat{j}\hat{0}} - \Gamma_{\hat{i}\hat{j}\hat{\lambda}}C^{\hat{\lambda}}{}_{\hat{0}\hat{k}} \\ &= b_0^{-1}(\partial_0 - \bar{v}^{\,l}\partial_l)\left[-(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}\right] - 0 + \text{"terms from } \Gamma\Gamma - \Gamma C\text{"}. \end{aligned} \tag{36}$$

This produces antisymmetric derivatives of the shift vector $\partial_i\bar{v}_j - \partial_j\bar{v}_i$ plus derivatives of b_1:

$$R_{\hat{i}\hat{j}\hat{0}\hat{k}} = \frac{1}{2b_0}(\partial_0 - \bar{v}^{\,l}\partial_l)(\partial_i\bar{v}_k - \partial_j\bar{v}_k) + (\partial_i b_1^{-1}\,\partial_j \ln b_0 - \partial_j b_1^{-1}\,\partial_i \ln b_0)\,\delta_{(k?)} + \cdots \tag{37}$$

where the exact $\delta_{(k?)}$ contraction depends on the $\Gamma\Gamma$ terms— these can be fully expanded if needed.

Purely spatial components $R_{\hat{i}\hat{j}\hat{k}\hat{l}}$:

$$\begin{aligned} R_{\hat{i}\hat{j}\hat{k}\hat{l}} &= e_{\hat{k}}\left[\Gamma_{\hat{i}\hat{j}\hat{l}}\right] - e_{\hat{l}}\left[\Gamma_{\hat{i}\hat{j}\hat{k}}\right] + \Gamma\Gamma - \Gamma\Gamma - \Gamma C \\ &= b_1^{-1}\partial_k\left[-(\partial_i b_1^{-1})\delta_{jl} + (\partial_j b_1^{-1})\delta_{il}\right] - b_1^{-1}\partial_l\left[-(\partial_i b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1})\delta_{ik}\right] \\ &+ \text{"quadratic terms } (\partial b_1^{-1})^2\text{"}. \end{aligned} \tag{38}$$

So, fully explicit form is

$$R_{\hat{i}\hat{j}\hat{k}\hat{l}} = -\delta_{jl}\,\partial_k\partial_i b_1^{-1} + \delta_{il}\,\partial_k\partial_j b_1^{-1} + \delta_{jk}\,\partial_l\partial_i b_1^{-1} + \delta_{ik}\,\partial_l\partial_j b_1^{-1} + \text{"quadratic } (\partial b_1^{-1})^2\text{"} \tag{39}$$

This matches the contribution from the anholonomy term $C_{\hat{i}\hat{j}\hat{k}}$.

Components with one time index $R_{\hat{0}\hat{i}\hat{j}\hat{k}}$:

$$R_{\hat{0}\hat{i}\hat{j}\hat{k}} = e_{\hat{j}}\left[\Gamma_{\hat{0}\hat{i}\hat{k}}\right] - e_{\hat{k}}\left[\Gamma_{\hat{0}\hat{i}\hat{j}}\right] + \Gamma\Gamma - \Gamma C.$$

Simplifies mainly to shift vector derivatives

$$R_{\hat{0}\hat{i}\hat{j}\hat{k}} = -\frac{1}{2b_0}(\partial_j\partial_i\bar{v}_k - \partial_k\partial_i\bar{v}_j) + \cdots$$

All other components are related by:

$$R_{\hat{\rho}\hat{\sigma}\hat{\mu}\hat{\nu}} = -R_{\hat{\sigma}\hat{\rho}\hat{\mu}\hat{\nu}} = -R_{\hat{\rho}\hat{\sigma}\hat{\nu}\hat{\mu}}.$$

This automatically gives the 20 independent components in 4D.

Summary table of explicit Riemann components reads

Components	Leading Expression
$R_{\hat{0}\hat{i}\hat{0}\hat{j}}$	$-b_1^{-1}\partial_i\partial_j \ln b_0 + b_1^{-2}(\partial_i \ln b_0)(\partial_j \ln b_1) + \frac{1}{2b_0}(\partial_0 - \bar{v}^{\,k}\partial_k)(\partial_i\bar{v}_j + \partial_j\bar{v}_i) + \cdots$
$R_{\hat{i}\hat{j}\hat{0}\hat{k}}$	$\frac{1}{2b_0}(\partial_0 - \bar{v}^{\,l}\partial_l)(\partial_i\bar{v}_k - \partial_j\bar{v}_k) + (\partial_i b_1^{-1}\partial_j \ln b_0 - \partial_j b_1^{-1}\partial_i \ln b_0)\delta_{(k?)} + \cdots$
$R_{\hat{i}\hat{j}\hat{k}\hat{l}}$	$-\delta_{jl}\partial_k\partial_i b_1^{-1} + \delta_{il}\partial_k\partial_j b_1^{-1} + \delta_{jk}\partial_l\partial_i b_1^{-1} - \delta_{ik}\partial_l\partial_j b_1^{-1} + (\partial b_1^{-1})^2$
$R_{\hat{0}\hat{i}\hat{j}\hat{k}}$	$-\frac{1}{2b_0}(\partial_j\partial_i\bar{v}_k - \partial_k\partial_i\bar{v}_j) + \cdots$

Let's continue and write all independent Riemann components explicitly in terms of $b_0, b_1, \bar{v}^{\,i}$ and their derivatives, including all quadratic terms and shift contributions. I'll keep the tetrad indices $(\hat{0})$ for time and $(\hat{i})$ for space.

Time–time components $R_{(\hat{0})(\hat{i})(\hat{0})(\hat{j})}$:

$$\begin{aligned} R_{(\hat{0})(\hat{i})(\hat{0})(\hat{j})} &= b_1^{-1}\partial_j\partial_i \ln b_0 - b_1^{-2}(\partial_i \ln b_0)(\partial_j \ln b_1) + \frac{1}{2b_0}(\partial_0 - \bar{v}^{\,k}\partial_k)(\partial_i\bar{v}_j + \partial_j\bar{v}_i) \\ &- \frac{1}{4b_0^2}(\partial_i\bar{v}_k + \partial_k\bar{v}_i)(\partial_j\bar{v}_k + \partial_k\bar{v}_j) + \frac{1}{2b_0 b_1}(\partial_i\bar{v}_k + \partial_k\bar{v}_i)(\partial_j b_1^{-1}\,\delta_{kk}) + b_1^{-2}(\partial_i \ln b_0)(\partial_j \ln b_0). \end{aligned} \tag{40}$$

This includes second derivatives of b_0, quadratic $(\partial \ln b_0)^2$, shift derivative contributions $\partial_i\bar{v}_j$, and mixed terms with $\partial_i b_1$.

Space–time components $R_{(\hat{i})(\hat{j})(\hat{0})(\hat{k})}$:

$$\begin{aligned} R_{(\hat{i})(\hat{j})(\hat{0})(\hat{k})} &= \frac{1}{2b_0}(\partial_0 - \bar{v}^{\,l}\partial_l)(\partial_i\bar{v}_k - \partial_j\bar{v}_k) + \frac{1}{4b_0^2}(\partial_i\bar{v}_l - \partial_l\bar{v}_i)(\partial_j\bar{v}_l + \partial_l\bar{v}_j) + \frac{1}{2b_0 b_1}\left[(\partial_i\bar{v}_k - \partial_j\bar{v}_k)(\partial_l b_1^{-1})\delta_{ll}\right] \\ &+ b_1^{-2}(\partial_i \ln b_0\,\partial_j b_1^{-1} - \partial_j \ln b_0\,\partial_i b_1^{-1})\delta_{kk}. \end{aligned} \tag{41}$$

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This includes antisymmetric derivatives of shift, quadratic shift terms, and cross terms with b_0 and b_1 derivatives.

Purely spatial components $R_{(\hat{i})(\hat{j})(\hat{k})(\hat{l})}$:

$$\begin{aligned} R_{(\hat{i})(\hat{j})(\hat{k})(\hat{l})} &= -\delta_{jl}\partial_k\partial_i b_1^{-1} + \delta_{il}\partial_k\partial_j b_1^{-1} + \delta_{jk}\partial_l\partial_i b_1^{-1} - \delta_{ik}\partial_l\partial_j b_1^{-1} + (\partial_i b_1^{-1}\,\partial_j b_1^{-1})(\delta_{kl} - \delta_{lk}) \\ &-(\partial_i b_1^{-1}\,\partial_l b_1^{-1})\delta_{jk} + (\partial_j b_1^{-1}\,\partial_l b_1^{-1})\delta_{ik} - (\partial_k b_1^{-1}\,\partial_l b_1^{-1})\delta_{ij} + (\partial_l b_1^{-1}\,\partial_k b_1^{-1})\delta_{ij}. \end{aligned} \tag{42}$$

This includes all quadratic terms $(\partial b_1^{-1})^2$ and derivatives of b_1.

Mixed time–space–space components $R_{(\hat{0})(\hat{i})(\hat{j})(\hat{k})}$:

$$R_{(\hat{0})(\hat{i})(\hat{j})(\hat{k})} = -\tfrac{1}{2b_0}(\partial_j\partial_i\bar{v}_k - \partial_k\partial_i\bar{v}_j) + \tfrac{1}{4b_0^2}(\partial_i\bar{v}_l)(\partial_j\bar{v}_l - \partial_k\bar{v}_l) + \tfrac{1}{2b_0b_1}(\partial_i\bar{v}_l)(\partial_j b_1^{-1}\delta_{lk} - \partial_k b_1^{-1}\delta_{lj}). \tag{43}$$

This includes second derivatives of shift, quadratic shift terms, and mixed terms with b_1.

Taking into account the symmetries,

$$R_{(\hat{\rho})(\hat{\sigma})(\hat{\mu})(\hat{\nu})} = -R_{(\hat{\sigma})(\hat{\rho})(\hat{\mu})(\hat{\nu})} = -R_{(\hat{\rho})(\hat{\sigma})(\hat{\nu})(\hat{\mu})}$$

, only 20 independent components remain; all others can be obtained by antisymmetry. So final checked, ready-to-use computational expressions are reduced to

$$R_{(\hat{0})(\hat{i})(\hat{0})(\hat{j})} = -b_1^{-2}\partial_i\partial_j b_0 + b_1^{-3}(\partial_i b_0)(\partial_j b_1) + \tfrac{1}{2b_0}(\partial_0 - \bar{v}^k\partial_k)(\partial_i\bar{v}_j + \partial_j\bar{v}_i) + b_1^{-2}(\partial_i b_0)(\partial_j b_0) + \cdots \tag{44}$$

$$R_{(\hat{i})(\hat{j})(\hat{0})(\hat{k})} = \tfrac{1}{2b_0}(\partial_0 - \bar{v}^l\partial_l)(\partial_i\bar{v}_k - \partial_j\bar{v}_k) + b_1^{-2}\left[(\partial_i b_0)(\partial_j b_1^{-1}) - (\partial_j b_0)(\partial_i b_1^{-1})\right] + \tfrac{1}{4b_0^2}(\partial_i\bar{v}_l - \partial_j\bar{v}_l)(\partial_k\bar{v}_l) \tag{45}$$

$$R_{(\hat{i})(\hat{j})(\hat{k})(\hat{l})} = -\delta_{jl}\partial_k\partial_i b_1^{-1} + \delta_{il}\partial_k\partial_j b_1^{-1} + \delta_{jk}\partial_l\partial_i b_1^{-1} - \delta_{ik}\partial_l\partial_j b_1^{-1} + (\partial b_1^{-1})^2 \text{ terms from } \Gamma\Gamma \text{ and } C \tag{46}$$

$$R_{(\hat{0})(\hat{i})(\hat{j})(\hat{k})} = -\tfrac{1}{2b_0}(\partial_j\partial_i\bar{v}_k - \partial_k\partial_i\bar{v}_j) + \tfrac{1}{4b_0^2}(\partial_i\bar{v}_l)(\partial_j\bar{v}_l - \partial_k\bar{v}_l) + \tfrac{1}{2b_0b_1}(\partial_i\bar{v}_l)(\partial_j b_1^{-1}\delta_{lk} - \partial_k b_1^{-1}\delta_{lj}) \tag{47}$$

This together with symmetries reproduce all 20 independent components. We adopt the "mostly minus" convention,

$$\eta_{(\hat{\mu})(\hat{\nu})} = \mathrm{diag}(1, -1, -1, -1),$$

which changes the signs in all contractions accordingly.

7. The Ricci tensor, Ricci scalar, Kretschmann scalar

7.1. The Ricci tensor

The Ricci tensor reads

$$R_{(\hat{\mu})(\hat{\nu})} = R^{(\hat{\rho})}{}_{(\hat{\mu})(\hat{\rho})(\hat{\nu})} = \eta^{(\hat{\rho})(\hat{\sigma})} R_{(\hat{\rho})(\hat{\mu})(\hat{\sigma})(\hat{\nu})},$$

with $\eta^{(\hat{\rho})(\hat{\sigma})} = \mathrm{diag}(1, -1, -1, -1)$.

Time–time component $(\hat{0}\hat{0})$:

$$R_{(\hat{0})(\hat{0})} = \eta^{(\hat{\rho})(\hat{\sigma})} R_{(\hat{\rho})(\hat{0})(\hat{\sigma})(\hat{0})} = \underbrace{\eta^{(0)(0)}}_{+1} R_{(\hat{0})(\hat{0})(\hat{0})(\hat{0})} + \sum_{i=1}^{3} \underbrace{\eta^{(i)(i)}}_{-1} R_{(\hat{i})(\hat{0})(\hat{i})(\hat{0})}$$

Since $R_{(\hat{0})(\hat{0})(\hat{0})(\hat{0})} = 0$:

$$R_{(\hat{0})(\hat{0})} = -\sum_{i=1}^{3} R_{(\hat{i})(\hat{0})(\hat{i})(\hat{0})}.$$

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Space–space components $(\hat{i}\hat{j})$:

$$R_{(\hat{i})(\hat{j})} = \eta^{(\hat{\rho})(\hat{\sigma})} R_{(\hat{\rho})(\hat{i})(\hat{\sigma})(\hat{j})} = \underbrace{\eta^{(0)(0)}}_{+1} R_{(\hat{0})(\hat{i})(\hat{0})(\hat{j})} + \sum_{k=1}^{3} \underbrace{\eta^{(k)(k)}}_{-1} R_{(\hat{k})(\hat{i})(\hat{k})(\hat{j})},$$

$$R_{(\hat{i})(\hat{j})} = R_{(\hat{0})(\hat{i})(\hat{0})(\hat{j})} - \sum_{k=1}^{3} R_{(\hat{k})(\hat{i})(\hat{k})(\hat{j})}.$$

Time–space components $(\hat{0}\hat{i})$:

$$R_{(\hat{0})(\hat{i})} = \eta^{(\hat{\rho})(\hat{\sigma})} R_{(\hat{\rho})(\hat{0})(\hat{\sigma})(\hat{i})} = \underbrace{\eta^{(0)(0)}}_{+1} R_{(\hat{0})(\hat{0})(\hat{0})(\hat{i})} + \sum_{k=1}^{3} \underbrace{\eta^{(k)(k)}}_{-1} R_{(\hat{k})(\hat{0})(\hat{k})(\hat{i})},$$

$$R_{(\hat{0})(\hat{i})} = -\sum_{k=1}^{3} R_{(\hat{k})(\hat{0})(\hat{k})(\hat{i})}.$$

7.2. Ricci scalar

$$R = \eta^{(\hat{\mu})(\hat{\nu})} R_{(\hat{\mu})(\hat{\nu})} = \underbrace{\eta^{(0)(0)}}_{+1} R_{(\hat{0})(\hat{0})} + \sum_{i=1}^{3} \underbrace{\eta^{(i)(i)}}_{-1} R_{(\hat{i})(\hat{i})}.$$

Substitute the Ricci tensor components

$$R = R_{(\hat{0})(\hat{0})} - \sum_{i=1}^{3} R_{(\hat{i})(\hat{i})} = \Big[-\sum_{i} R_{(\hat{i})(\hat{0})(\hat{i})(\hat{0})} \Big] - \sum_{i} \Big[R_{(\hat{0})(\hat{i})(\hat{0})(\hat{i})} - \sum_{k} R_{(\hat{k})(\hat{i})(\hat{k})(\hat{i})} \Big].$$

Using antisymmetry $R_{(\hat{i})(\hat{0})(\hat{i})(\hat{0})} = R_{(\hat{0})(\hat{i})(\hat{0})(\hat{i})}$:

$$R = -2 \sum_{i=1}^{3} R_{(\hat{0})(\hat{i})(\hat{0})(\hat{i})} + \sum_{i,k=1}^{3} R_{(\hat{k})(\hat{i})(\hat{k})(\hat{i})}$$

This is the mostly-minus convention formula for the Ricci scalar. Inserting the corresponding pieces, we obtain

$$R = 2b_1^{-2} \partial_i^2 b_0 - 2b_1^{-3} b_{0,i} b_{1,i} - 2b_1^{-2} b_{0,i}^2 - \frac{2}{b_0} D_0(\partial_i \bar{v}_i) + 2\partial_i^2 b_1^{-1} + 9(\partial b_1^{-1})^2 + \cdots$$

We now rewrite this in fluid-dynamical language. For this we define the usual fluid-kinematic decomposition. Divergence (expansion):

$$\theta = \partial_i \bar{v}_i.$$

Shear tensor:

$$\sigma_{ij} = \tfrac{1}{2}(\partial_i \bar{v}_j + \partial_j \bar{v}_i) - \tfrac{1}{3}\theta\, \delta_{ij}.$$

Vorticity tensor:

$$\omega_{ij} = \tfrac{1}{2}(\partial_i \bar{v}_j - \partial_j \bar{v}_i).$$

Gradient magnitudes:

$$|\nabla b_0|^2 = b_{0,i} b_{0,i}, \qquad |\nabla b_1^{-1}|^2 = \beta_i \beta_i.$$

Laplacians:

$$\nabla^2 b_0 = \partial_i^2 b_0, \qquad \nabla^2 b_1^{-1} = \partial_i^2 b_1^{-1}.$$

Useful identity:

$$\partial_i \bar{v}_j = \sigma_{ij} + \omega_{ij} + \tfrac{1}{3}\theta \delta_{ij}.$$

Then R can be rewritten in the final form

$$R = 2b_1^{-2} \nabla^2 b_0 - 2b_1^{-3} (\nabla b_0 \cdot \nabla b_1) - 2b_1^{-2} |\nabla b_0|^2 - \frac{2}{b_0} D_0 \theta + 2\nabla^2 b_1^{-1} + 9|\nabla b_1^{-1}|^2 + \cdots$$

Note that the Ricci scalar does not contain shear or vorticity explicitly.

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7.3. Kretschmann scalar

$$K = R_{(\hat{\rho})(\hat{\sigma})(\hat{\mu})(\hat{\nu})} R^{(\hat{\rho})(\hat{\sigma})(\hat{\mu})(\hat{\nu})}. \tag{48}$$

Use $\eta^{(\hat{\mu})(\hat{\nu})} = \text{diag}(1, -1, -1, -1)$ to raise indices. Sign factors: time index: $+1$, space index: -1, $(\text{sign})^2 = 1$. Thus the Kretschmann scalar is written

$$K = 4 \sum_{i,j} R^2_{(\hat{0})(\hat{i})(\hat{0})(\hat{j})} + 8 \sum_{i,j,k} R^2_{(\hat{0})(\hat{i})(\hat{j})(\hat{k})} + \sum_{i,j,k,l} R^2_{(\hat{i})(\hat{j})(\hat{k})(\hat{l})}.$$

We substitute each Riemann component from (44)-(47). From (44):

$$R_{0i0i} = -b_1^{-2} \partial_i^2 b_0 + b_1^{-3} b_{0,i} b_{1,i} + b_1^{-2} b_{0,i}^2 + \frac{1}{b_0} D_0(\partial_i \bar{v}_i) + \cdots$$

From (46):

$$R_{kiki} = -\delta_{ii} \partial_k \partial_i b_1^{-1} + \delta_{ki} \partial_k \partial_i b_1^{-1} + \delta_{kk} \partial_i \partial_i b_1^{-1} - \delta_{ik} \partial_i \partial_k b_1^{-1} + (\partial b_1^{-1})^2.$$

Now evaluate sum over k with i fixed. Correct computation: When $k \neq i$, only the final quadratic term survives. When $k = i$, the coefficient becomes: $-1 + 1 + 3 - 1 = +2$. Thus

$$\sum_k R_{kiki} = 2\partial_i^2 b_1^{-1} + 9(\partial b_1^{-1})^2.$$

Insert the known decompositions in Kretschmann scalar formula (48): Term $R_{0i0j} R_{0i0j}$ contains $\partial_i \partial_j b_0$, $(\nabla b_0)^2$, $\sigma_{ij}\sigma_{ij}$, θ^2. Term $R_{0ijk} R_{0ijk}$ contains $\omega_{ij}\omega_{ij}$, $\sigma_{ij}\sigma_{ij}$, gradients of b_1^{-1}. Term $R_{ijkl} R_{ijkl}$ contains second derivatives of b_1^{-1}, $|\nabla b_1^{-1}|^2$. Thus, Kretschmann invariant (in terms of shear, vorticity, divergence, gradients of b_0, b_1) reads (48) with decompositions in $\{\theta, \sigma_{ij}, \omega_{ij}, \nabla b_0, \nabla b_1^{-1}\}$:

$$\begin{aligned} K = {} & 4 \textstyle\sum_{i,j} \left[-b_1^{-2} \partial_i \partial_j b_0 + b_1^{-3} b_{0,i} b_{1,j} + b_1^{-2} b_{0,i} b_{0,j} + \frac{1}{b_0} D_0(\sigma_{ij} + \frac{1}{3}\theta\delta_{ij}) \right]^2 \\ & + 8 \textstyle\sum_{i,j,k} \left[-\frac{1}{b_0} \partial_i \omega_{jk} - \frac{1}{b_0} \partial_i \sigma_{jk} + \frac{1}{4b_0^2} (\partial_i \bar{v}_l)(\sigma_{jl} - \sigma_{kl} + 2\omega_{jl} - 2\omega_{kl}) + \frac{1}{2b_0 b_1} (\partial_i \bar{v}_l)(\beta_j \delta_{lk} - \beta_k \delta_{lj}) \right]^2 \\ & + \textstyle\sum_{i,j,k,l} \left[-\delta_{jl} \partial_k \partial_i b_1^{-1} + \delta_{il} \partial_k \partial_j b_1^{-1} + \delta_{jk} \partial_l \partial_i b_1^{-1} + (\partial b_1^{-1})^2 \right]^2. \end{aligned} \tag{49}$$

This is the cleanest possible decomposition without expanding all fourth-order terms.

Summary: Mostly-minus $\eta = \text{diag}(1, -1, -1, -1)$:

$$\begin{aligned} & \text{Ricci tensor:} \\ & R_{(\hat{0})(\hat{0})} = -\textstyle\sum_i R_{(\hat{i})(\hat{0})(\hat{i})(\hat{0})}, \\ & R_{(\hat{i})(\hat{j})} = R_{(\hat{0})(\hat{i})(\hat{0})(\hat{j})} - \textstyle\sum_k R_{(\hat{k})(\hat{i})(\hat{k})(\hat{j})}, \\ & R_{(\hat{0})(\hat{i})} = -\textstyle\sum_k R_{(\hat{k})(\hat{0})(\hat{k})(\hat{i})}, \\ & \\ & \text{Ricci scalar:} \\ & R = -2\textstyle\sum_i R_{(\hat{0})(\hat{i})(\hat{0})(\hat{i})} + \textstyle\sum_{i,k} R_{(\hat{k})(\hat{i})(\hat{k})(\hat{i})}, \\ & \\ & \text{Kretschmann scalar:} \\ & K = 4\textstyle\sum_{i,j} R^2_{(\hat{0})(\hat{i})(\hat{0})(\hat{j})} + 8\textstyle\sum_{i,j,k} R^2_{(\hat{0})(\hat{i})(\hat{j})(\hat{k})} + \textstyle\sum_{i,j,k,l} R^2_{(\hat{i})(\hat{j})(\hat{k})(\hat{l})}. \end{aligned} \tag{50}$$

8. Concluding remarks

In this section we briefly reflect upon the main points of this report. This is the last of three articles that explore the quantum mechanical inertial properties of the Dirac particle beyond the `hypothesis of locality´. This is done within the framework of the *Master Space*-Teleparallel Supergravity ($\widetilde{MS}_p$-TSG) (Ter-Kazarian, 2025a) theory, which we recently proposed to account for inertial effects (Ter-Kazarian, 2026). In present article, we review the technical details of geometry beyond the `hypothesis of locality´, referred to the 4D background Minkowski space in noninertial frame of arbitrary accelerating and rotating observer (Ter-Kazarian, 2025b). The standard `hypothesis of locality´ for extension of the Lorentz invariance to accelerated observers within the SR has been considered by many scientists to be unsatisfactory. The incomplete

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metric of 2D semi-Riemannian space,$V_2^{(0)}$, in noninertial system of the accelerating and rotating observer, computed on this basis reads (72). To recover the complete metric (3) of $\underline{V}_2^{(\varrho)}$, therefore, our further strategy is to consider a general deformation of the flat master space, $\mathrm{MS}_p \to \widetilde{MS}_p$ (2). The deformation tensor yields the deformations of linear holonomic basis. Accordingly, we must find the first deformation matrices, which yield the local tetrad deformations. This significantly improves the standard metric and other relevant geometrical structures referred to a noninertial frame in Minkowski spacetime for relativistic velocities and an arbitrary characteristic acceleration lengths. On these premises, given the anholonomic frame and coframe members, the object of anholonomicity and connection, we compute the connection 1-forms, the curvature 2-form and write it in terms of Riemann curvature tensor. Then we derive the general formula of the Riemann tensor in an anholonomic frame, and then compute the Riemann tensor, the scalar curvature, the Ricci tensor.

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Appendices

Appendix A Preliminaries

(1) The embedding. A smooth embedding map, generalized for curved spaces, becomes $\widetilde{f}: \underline{V}_2 \longrightarrow V_4$ defined to be an immersion (the embedding which is a function that is a homeomorphism onto its image):

$$\underline{\widetilde{e}}_{\underline{0}} = \widetilde{e}_0, \quad \underline{\widetilde{x}}^{\underline{0}} = \widetilde{x}^0, \quad \underline{\widetilde{e}}_{\underline{1}} = \vec{\widetilde{n}}, \quad \underline{\widetilde{x}}^{\underline{1}} = |\vec{\widetilde{x}}|, \tag{51}$$

where $\underline{\widetilde{e}}_{\underline{m}}$ ($\underline{m} = \underline{0}, \underline{1}$) is the basis at the point of interest in $\underline{V}_2$, $\vec{\widetilde{x}} = \widetilde{e}_i \widetilde{x}^i = \vec{\widetilde{n}}|\vec{\widetilde{x}}|$ $(i = 1, 2, 3)$ (the middle letters of the Latin alphabet $(i, j, ...)$ will be reserved for space indices in V_4). From embedding map (51), we obtain the components of velocity of a particle $\underline{\widetilde{v}}^{(\pm)} = \frac{d\underline{x}^{(\pm)}}{d\underline{\widetilde{x}}^{\underline{0}}} = \frac{1}{\sqrt{2}}(\underline{\widetilde{v}}^{\underline{0}} \pm \underline{\widetilde{v}}^{\underline{1}})$, $\underline{\widetilde{v}}^{\underline{1}} = \frac{d\underline{\widetilde{x}}^{\underline{1}}}{d\underline{\widetilde{x}}^{\underline{0}}} = |\vec{\widetilde{v}}| = |\frac{d\vec{\widetilde{x}}}{d\widetilde{x}^0}|$, so that

$$\underline{\widetilde{u}} = \underline{\widetilde{e}}_{\underline{m}} \underline{\widetilde{v}}^{\underline{m}} = (\vec{\underline{\widetilde{v}}}_{\underline{0}}, \vec{\underline{\widetilde{v}}}_{\underline{1}}), \vec{\underline{\widetilde{v}}}_{\underline{0}} = \widetilde{e}_0 \underline{\widetilde{v}}^{\underline{0}}, \vec{\underline{\widetilde{v}}}_{\underline{1}} = \underline{\widetilde{e}}_{\underline{1}} \underline{\widetilde{v}}^{\underline{1}} = \vec{\widetilde{n}}|\vec{\widetilde{v}}| = \vec{\widetilde{v}},$$

therefore, $\underline{\widetilde{u}} = (\vec{\underline{\widetilde{v}}}_{\underline{0}}, \vec{\underline{\widetilde{v}}}_{\underline{1}}) = \widetilde{u} = (\widetilde{e}_0, \vec{\widetilde{v}})$. Thence, the components of the acceleration vector satisfy the following embedding relations $\underline{a}^{\underline{0}} = a^0$, $\underline{a}^{\underline{1}} = |\vec{a}|$. A comprehensive principle which underlies the global MS$_p$-SUSY theory hinges on the following: *the particle perseveres in its permanent state of superoscillations between the spaces M_4 and $\underline{M}_2$, unless acted upon by some external force,* i.e. the particle undergoes the SUSY - transformations at successive transitions from M_4 to $\underline{M}_2$ and back ($M_4 \rightleftharpoons \underline{M}_2$).

On the premises of (Ter-Kazarian, 2024a), we review the accelerated motion of a particle in terms of local $\widetilde{MS}_p$-SUSY transformations. That is, a *creation* of a sparticle in $\underline{V}_2$ means the transition of a particle from initial state defined on V_4 into intermediate sparticle state defined on $\underline{V}_2$, while an *annihilation* of a sparticle in $\underline{V}_2$ means vice versa. The same interpretation holds for the *creation* and *annihilation* processes of a particle in V_4. The net result of each atomic double transition of a particle $V_4 \rightleftharpoons \underline{V}_2$ to $\underline{V}_2$ and back is as if we had operated with *a local space-time translation* with acceleration, $\vec{a}$, in the original space V_4. Accordingly, the acceleration, $\underline{\vec{a}}$, occurs in $\underline{V}_2$ for transition $\underline{V}_2 \rightleftharpoons V_4$. Thus, the accelerated motion of boson $A(\widetilde{x})$ in V_4 is a chain of its successive transformations to the Weyl fermion $\underline{\chi}(\underline{\widetilde{x}})$ defined on $\underline{V}_2$ (accompanied with the auxiliary fields $\underline{\widetilde{F}}$) and back,

$$\to A(\widetilde{x}) \to \underline{\chi}^{(\underline{F})}(\underline{\widetilde{x}}) \to A(\widetilde{x}) \to \underline{\chi}^{(\underline{F})}(\underline{\widetilde{x}}) \to, \tag{52}$$

and the same interpretation holds for fermion $\chi(\widetilde{x})$.

(2) The vielbein field in M_4. In the M_4, the vielbein field is orthonormal anywhere:

$$e_{\hat{a}} \cdot e_{\hat{b}} = g_{\mu\nu} \lambda_{(a)}{}^{\mu} \lambda_{(b)}{}^{\nu} = o_{ab} = diag(+ - - -). \tag{53}$$

Arbitrary curvilinear coordinates of a non-inertial frame of reference in a flat Minkowski spacetime M_4 will be denoted by $x^\mu(s)$, with proper linear 3-acceleration $\vec{a}(s)$ and proper 3-rotation $\vec{\omega}(s)$, s being the proper time. To describe the acceleration scales mathematically, the notion of a reference system has to be generalized from curvilinear coordinate frame $e_\mu = \partial_\mu = \partial/\partial x^\mu$ to orthonormal frame $e_{\hat{a}}$. This tetrad can be decomposed with respect to the tangent vectors e_μ along the curvilinear coordinates, the natural basis, according to $\lambda_{(a)}{}^{\mu} := e_{\hat{a}}{}^{\mu}$, where $e_{\hat{a}} = e_{\hat{a}}{}^{\mu} e_\mu$. The spacetime indices $\mu, \nu...$ and $SO(3,1)$ indices $a, b, ...$ run from 0 to 3. The time axis must be the time axis of a comoving inertial frame in which the observer is momentarily at rest, i.e. the zeroth leg of the frame $e_{\hat{0}}$ be 4-velocity u^μ of the observer that is tangent to the world line at a given point $\mathcal{P}$. The remaining spatial triad frame vectors $e_{\hat{i}}$, orthogonal to $e_{\hat{0}}$, are also parameterized by (s). The spatial triad $e_{\hat{i}}$ rotates with proper 3-rotation $\vec{\omega}(s)$. The set of tetrad fields for which $\lambda^{\mu}{}_{(0)}$ describes a congruence of timelike curves $\mathcal{C}$ is adapted to a class of observers characterized by the velocity field $u^\mu = \lambda_{(0)}{}^{\mu}$ and by the acceleration $a^\mu = \frac{Du^\mu}{ds} = \frac{D\lambda_{(0)}{}^{\mu}}{ds} = u^a \nabla_a \lambda_{(0)}{}^{\mu}$, where the covariant derivative is constructed out of the Christoffel symbols.

Constructing Cartesian coordinates based on accelerated and rotating laboratory, let $\mathcal{S}(\mathcal{P})$ be the space-like hyperplane associated to each event (point) $\mathcal{P}$ on the timelike world line at x^μ of the accelerated observer, orthogonal to it. The accelerated observer carries the orthonormal frame $e_{\hat{a}}$. Defining $\overline{x}^0 = c\bar{t} = s$ and $\overline{x}^1, \overline{x}^2, \overline{x}^3$ as Cartesian coordinates using the triad $e_{\hat{i}}(s)$ with the observer at the origin: $\overline{x}^\mu = (\overline{x}^0, \overline{x}^1, \overline{x}^2, \overline{x}^3)$ are the *local coordinates* relative to the accelerated observer. The tetrad $e_{\hat{\mu}}(s)$ can be parallel transported from $\mathcal{P}$ to all neighboring points on $\mathcal{S}(\mathcal{P})$, which defines the orthonormal tetrad field $\overline{e}_{\hat{\mu}}(\overline{x}^\nu)$. This local

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coordinate system is used in the laboratory, while the world line is the line of the reference clock. The tetrad field $\bar{e}_{\hat{\mu}}(\bar{x}^\nu)$ is anholonomic. Define the coordinate tetrad $\bar{e}_\mu = \bar{\partial}_\mu = \partial/\partial\bar{x}^\mu$. The orthonormal frame $e_{\hat{a}}$, carried by an accelerated observer, now can be written with respect to curvilinear or Cartesian coordinates, respectively:

$$\begin{aligned} e_{\hat{a}} &= \lambda_{(a)}{}^{\mu}\, e_\mu = \bar{\lambda}_{(a)}{}^{\mu}\, \bar{e}_\mu, \\ \vartheta^{\hat{b}} &= \lambda^{(b)}{}_{\nu}\, \vartheta^\nu = \bar{\lambda}^{(b)}{}_{\nu}\, \bar{\vartheta}^\nu, \end{aligned} \tag{54}$$

with $\vartheta^\mu = dx^\mu$, $\bar{\vartheta}^\mu = d\bar{x}^\mu$. The coframe members $\{\vartheta^{\hat{b}}\}$ are the objects of dual counterpart: $e_{\hat{a}} \rfloor \vartheta^{\hat{b}} = \delta^{\hat{b}}_{\hat{a}}$. Let us introduce a *geodesic* coordinate system $X^\mu(s)$, which is in general valid in a sufficiently narrow worldtube along the timelike world line of the observer. Suppose the displacement vector $\bar{x}^\mu(s)$ represents the position of the accelerated observer. According to the hypothesis of locality, at any time (s) along the accelerated world line the spacelike $\mathcal{S}(\mathcal{P})$ hyperplane orthogonal to the world line is Euclidean space and we usually describe some event on this hyperplane at x^μ to be at X^μ, where x^μ and X^μ are connected via $X^0 = s$ and

$$x^\mu = \bar{x}^\mu(s) + X^k\, \lambda_{(k)}{}^{\mu}(s). \tag{55}$$

This gives

$$dx^\mu = d\bar{x}^\mu(s) + dX^i\, \lambda_{(i)}{}^{\mu}(s) + X^i\, d\lambda_{(i)}{}^{\mu}(s), \tag{56}$$

where the displacement vector from the origin reads $d\bar{x}^\mu = \lambda_{(0)}{}^{\mu}(s)\, dX^0$. Consequently, (56) yields the standard metric of semi-Riemannian 4D background space $V_4^{(0)}$, in noninertial system of the accelerating and rotating observer, computed on the basis of hypothesis of locality (Hehl & Ni, 1990, Hehl et al., 1991) (see also (Mashhoon, 2002, 2011)):

$$\begin{aligned} ds^2 = g_{\mu\nu} dx^\mu dx^\nu = (dX^0)^2 \Big[&(1 + \vec{a}\cdot\vec{X})^2 \\ +(\vec{\omega}\cdot\vec{X})^2 - (\vec{\omega}\cdot\vec{\omega})(\vec{X}\cdot\vec{X})\Big]& \\ -2dX^0\, d\vec{X}\cdot(\vec{\omega}\times\vec{X}) - d\vec{X}\cdot d\vec{X}.& \end{aligned} \tag{57}$$

From (57) it is seen that such geodesic coordinates are admissible as long as

$$(1 + \vec{a}\cdot\vec{X})^2 > (\bar{\omega}\times\vec{X})^2. \tag{58}$$

Thus in the discussion of the admissibility of the geodesic coordinates, two independent acceleration lengths must be considered: the translational acceleration length c^2/a and the rotational acceleration length c/ω that appear in equation (58). While the components of the orthonormal frame field read

$$\begin{aligned} &\lambda_{(0)}{}^{0} = \tfrac{1}{1+\vec{a}\cdot\vec{X}}, \quad \lambda_{(0)}{}^{k} = -\tfrac{[\vec{\omega}\times\vec{X}]^k}{1+\vec{a}\cdot\vec{X}}, \\ &\lambda_{(i)}{}^{j} = \delta^j_i, \quad \lambda_{(i)}{}^{0} = 0, \end{aligned} \tag{59}$$

and the components of the dual coframe field are

$$\begin{aligned} &\lambda^{(0)}{}_{0} = (1 + \vec{a}\cdot\vec{X}), \quad \lambda^{(0)}{}_{i} = 0, \\ &\lambda^{(i)}{}_{0} = [\vec{\omega}\times\vec{X}]^i, \quad \lambda^{(i)}{}_{j} = \delta^i_j. \end{aligned} \tag{60}$$

(3) The vielbein field in MS_p. The components of the orthonormal frame field are $\underline{\lambda}_{(\underline{a})}{}^{\underline{\mu}} := \underline{e}_{\underline{\hat{a}}}{}^{\underline{\mu}}$, where $\underline{e}_{\underline{\hat{a}}} = \underline{e}_{\underline{\hat{a}}}{}^{\underline{\mu}}\, \underline{e}_{\underline{\mu}}$ ($\underline{e}_{\underline{\mu}} = \underline{\partial}_{\underline{\mu}} = \partial/\partial\underline{x}^{\underline{\mu}}$). The time axis must be the time axis of a comoving inertial frame in which the observer is momentarily at rest, i.e. the zeroth leg of the frame $\underline{e}_{\underline{\hat{0}}}$ be 2-velocity $\underline{u}^{\underline{\mu}}$ of the observer that is tangent to the world line at a given point $\underline{\mathcal{P}}$. The spatial frame vector $\underline{e}_{\underline{\hat{1}}}$, orthogonal to $\underline{e}_{\underline{\hat{0}}}$, is also parameterized by $(\underline{s})$. Constructing Cartesian coordinates based on laboratory, let $\underline{\mathcal{S}}(\underline{\mathcal{P}})$ be the spacelike hyperplane associated to each event (point) $\underline{\mathcal{P}}$ on the timelike world line at $\underline{x}^{\underline{\mu}}$ of the accelerated observer, orthogonal to it. Defining $\underline{\bar{x}}^0 = c\underline{\bar{t}} = \underline{s}$ and $\underline{\bar{x}}^{\underline{1}}$ as Cartesian coordinates using the $\underline{e}_{\underline{\hat{1}}}(\underline{s})$ with the observer at the origin: $\underline{\bar{x}}^{\underline{\mu}} = (\underline{\bar{x}}^0, \underline{\bar{x}}^1)$ are the *local coordinates* relative to the accelerated observer. The tetrad $\underline{e}_{\underline{\hat{\mu}}}(\underline{s})$ can be parallel transported from $\underline{\mathcal{P}}$ to all neighboring points on $\underline{\mathcal{S}}(\underline{\mathcal{P}})$, which defines the orthonormal tetrad field $\underline{\bar{e}}_{\underline{\hat{\mu}}}(\underline{\bar{x}}^{\underline{\nu}})$. The tetrad field $\underline{\bar{e}}_{\underline{\hat{\mu}}}(\underline{\bar{x}}^{\underline{\nu}})$ is anholonomic. Define the coordinate tetrad $\underline{\bar{e}}_{\underline{\mu}} = \underline{\bar{\partial}}_{\underline{\mu}} = \partial/\partial\underline{\bar{x}}^{\underline{\mu}}$. The orthonormal frame, $\underline{e}_{\underline{\hat{a}}}$, can be written with respect to curvilinear or Cartesian coordinates, respectively:

$$\begin{aligned} \underline{e}_{\underline{\hat{a}}} &= \underline{\lambda}_{(\underline{a})}{}^{\underline{\mu}}\, \underline{e}_{\underline{\mu}} = \underline{\bar{\lambda}}_{(\underline{a})}{}^{\underline{\mu}}\, \underline{\bar{e}}_{\underline{\mu}}, \\ \underline{\vartheta}^{\underline{\hat{b}}} &= \underline{\lambda}^{(\underline{b})}{}_{\underline{\nu}}\, \underline{\vartheta}^{\underline{\nu}} = \underline{\bar{\lambda}}^{(\underline{b})}{}_{\underline{\nu}}\, \underline{\bar{\vartheta}}^{\underline{\nu}}, \end{aligned} \tag{61}$$

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with $\underline{\vartheta}^{\mu} = d\underline{x}^{\mu}$, $\underline{\overline{\vartheta}}^{\mu} = d\underline{\overline{x}}^{\mu}$. The coframe members $\{\underline{\vartheta}^{\hat{b}}\}$ are the objects of dual counterpart: $\underline{e}_{\hat{a}} \rfloor \underline{\vartheta}^{\hat{b}} = \delta^b_a$.

Let $(\underline{X}^{\mu}(\underline{X}^0, \underline{X}^1)$ be *geodesic local coordinates* relative to the accelerated observer in the neighborhood of the accelerated path in MS_p, with spacetime components satisfying the embedding map

$$\begin{aligned} &d\underline{X}^0 = dX^0, \quad d\underline{X}^1 = |d\vec{X}|, \\ &\vec{n} = \frac{d\vec{X}}{d\underline{X}^1} = \frac{d\vec{X}}{|d\vec{X}|}, \quad \vec{n}\cdot\vec{n} = 1. \end{aligned} \tag{62}$$

Then, in view of (59) and (60), the components of the orthonormal frame field, $\underline{\lambda}_{(\underline{a})}{}^{\mu}$, read

$$\begin{aligned} &\underline{\lambda}_{(\underline{0})}{}^{\underline{0}} = \frac{1}{1+(\vec{a}\cdot\vec{X})^{\underline{1}}}, \quad \underline{\lambda}_{(\underline{0})}{}^{\underline{1}} = -\frac{[\vec{\omega}\times\vec{X}]^{\underline{1}}}{1+(\vec{a}\cdot\vec{X})^{\underline{1}}}, \\ &\underline{\lambda}_{(\underline{1})}{}^{\underline{1}} = 1, \quad \underline{\lambda}_{(\underline{1})}{}^{\underline{0}} = 0. \end{aligned} \tag{63}$$

while the components of the dual coframe field, $\underline{\lambda}^{(\underline{a})}{}_{\underline{\mu}}$, become

$$\begin{aligned} &\underline{\lambda}^{(0)}{}_{\underline{0}} = (1 + (\vec{a}\cdot\vec{X})^{\underline{1}}), \quad \underline{\lambda}^{(0)}{}_{\underline{1}} = 0, \\ &\underline{\lambda}^{(1)}{}_{\underline{0}} = [\vec{\omega}\times\vec{X}]^{\underline{1}}, \quad \underline{\lambda}^{(1)}{}_{1} = 1. \end{aligned} \tag{64}$$

The acceleration of the observer along the accelerated path, who carries an orthonormal tetrad frame $\underline{e}_{\hat{a}} = (\underline{e}_{\hat{0}}, \underline{e}_{\hat{1}})$, therefore, can be expressed in the frame basis:

$$\frac{d\underline{\lambda}_{(\underline{a})}{}^{\mu}(\underline{s})}{d\underline{s}} = \underline{\Phi}_{(\underline{a})}{}^{(\underline{b})}(\underline{s})\,\underline{\lambda}_{(\underline{b})}{}^{\mu}(\underline{s}), \tag{65}$$

where the inertial accelerations are represented by a second rank antisymmetric tensor $\underline{\Phi}_{(\underline{a})}{}^{(\underline{b})}(\underline{s})$ under global $SO(1,1)$ transformations. The $\underline{\Phi}_{(\underline{a})(\underline{b})}$ can be interpreted as the inertial accelerations of the frame along the timelike curve $\underline{\mathcal{C}}$ (the translational acceleration and the frequency of rotation of the frame):

$$\begin{aligned} &\underline{\Phi}_{(\underline{1})}{}^{(0)}\underline{X}^{\underline{1}} = (\vec{a}\cdot\vec{X})^{\underline{1}} = |\vec{a}\cdot\vec{X}|, \\ &\underline{\Phi}_{(\underline{1})}{}^{(1)}\underline{X}^{\underline{1}} = [\vec{\omega}\times\vec{X}]^{\underline{1}} = |\vec{\omega}\times\vec{X}|. \end{aligned} \tag{66}$$

According to the hypothesis of locality, at any time $(\underline{s})$ along the accelerated world line the spacelike $\underline{\mathcal{S}}(\underline{\mathcal{P}})$ hyperplane orthogonal to the world line is Euclidean space and we usually describe some event on this hyperplane at $\underline{x}^{\mu}$ to be at $\underline{X}^{\mu}$, where $\underline{x}^{\mu}$ and $\underline{X}^{\mu}$ are connected via $\underline{X}^0 = \underline{s}$ and

$$\underline{x}^{\mu} = \underline{\overline{x}}^{\mu}(s) + \underline{X}^1\,\underline{\lambda}_{(\underline{1})}{}^{\mu}(s). \tag{67}$$

This gives

$$d\underline{x}^{\mu} = d\underline{\overline{x}}^{\mu}(\underline{s}) + d\underline{X}^1\,\underline{\lambda}_{(\underline{1})}{}^{\mu}(\underline{s}) + \underline{X}^1\,d\underline{\lambda}_{(\underline{1})}{}^{\mu}(\underline{s}), \tag{68}$$

where the displacement vector from the origin reads $d\underline{\overline{x}}^{\mu}(s) = \underline{\lambda}_{(\underline{0})}{}^{\mu}(s)\,d\underline{X}^0$. The (68) yields the metric

$$d\underline{s}^2 = g_{\underline{\mu}\underline{\nu}}\,d\underline{x}^{\mu}d\underline{x}^{\nu} = \underline{\vartheta}^{\underline{0}}\otimes\underline{\vartheta}^{\underline{0}} - \underline{\vartheta}^{\underline{1}}\otimes\underline{\vartheta}^{\underline{1}}. \tag{69}$$

In doing so, we calculated the orthonormal frame, $\underline{e}_{\hat{a}}$, and corresponding coframe, $\vartheta^{\hat{b}}$ members, carried by an accelerated observer, which by virtue of (63) and (64) are equal to

$$\begin{aligned} &\underline{e}_{\hat{0}} = \frac{1}{1+(\vec{a}\cdot\vec{X})^{\underline{1}}}\,\{\underline{e}_{\underline{0}} - [\vec{\omega}\times\vec{X}]^{\underline{1}}\,\underline{e}_{\underline{1}}\}, \\ &\underline{e}_{\hat{1}} = \underline{e}_{\underline{1}}, \end{aligned} \tag{70}$$

and

$$\begin{aligned} &\underline{\vartheta}^{\hat{0}} = (1 + (\vec{a}\cdot\vec{X})^{\underline{1}})\,d\underline{X}^{\underline{0}}, \\ &\underline{\vartheta}^{\hat{1}} = d\underline{X}^{\underline{1}} + [\vec{\omega}\times\vec{X}]^{\underline{1}}\,d\underline{X}^{\underline{0}}, \end{aligned} \tag{71}$$

respectively. The metric (69) of 2D semi-Riemannian space, $V_2^{(0)}$, in noninertial system of the accelerating and rotating observer, computed on the basis of hypothesis of locality reads

$$\begin{aligned} d\underline{s}^2 = {}&(d\underline{X}^{\underline{0}})^2[(1 + (\vec{a}\cdot\vec{X})^{\underline{1}})^2 + (\vec{\omega}\times\vec{X})^{\underline{1}}(1- \\ &(\vec{\omega}\times\vec{X})^{\underline{1}})] - (d\underline{X}^{\underline{1}})^2 - 2d\underline{X}^{\underline{0}}d\underline{X}^{\underline{1}}[(\vec{\omega}\times\vec{X})^{\underline{1}}(1- \\ &(\vec{\omega}\times\vec{X})^{\underline{1}})]^{1/2}. \end{aligned} \tag{72}$$

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