Non-Linear Realizations of 'Distortion' Group

G.Ter-Kazarian*

Byurakan Astrophysical Observatory, Byurakan, Aragatsotn Province, Armenia

Abstract

In the framework of the method of phenomenological Lagrangians, we address the non-linear realizations of the Lie group G concerning the 'distortion' of 12-dimensional space. Treating the 'distortion' group G and its stationary subgroup $H = SO(3)$, respectively, as the dynamical group and its algebraic subgroup, we identify the parameters of the quotient space G/H of adjacent classes by the subgroup H with the Goldstone fields of spontaneous breaking of 'distortion' symmetry. We study the geometrical structure of the space of parameters in terms of Cartan forms introduced through the appropriate Maurer-Cartan's structure equations. This allows us to infer the group invariants and calculate the conserved currents as its inevitable corollaries.

Keywords: *Phenomenological Lagrangians–non-linear realizations of Lie group–Symmetry breaking–Maurer-Cartan's equations–Conserved currents*

1. Introduction

We have gained some insight into the concept of gauge group of gravitation in the proposed 'general gauge principle' (GGP) formalism Ter-Kazarian (1997, 2001). This is an alternative to the conventional gauge principle and accounts for gravity generated by hidden local internal symmetries implemented on 'maximally symmetric' (MS) auxiliary space. This formalism manifests its practical and technical virtue, providing a framework for the unitary mapping of the conventional gauge field dynamics from auxiliary MS-space to curved Riemannian space. In this, we need only to regard the auxiliary MS-space with a whole set of well-defined Killing vectors as a guide in dealing with an intricate 'jungle' of curved Riemannian geometry. On the other hand, this formalism allows for a subsequent extension of the curvature of the space-time continuum to the 'distortion' of 12-dimensional MS-space: $G_{12} = G_6 \oplus \overline{G}_6$, $DimG_{12} = 12$, $DimG_6 = 6$, $Dim\overline{G}_6 = 6$. As that initial phase of the investigation neared completion, however, it also became apparent that the most controversial point of the suggested approach is that 'distortion' transformations are not uniquely specified, leaving open the possibility to assign many physically inequivalent metrics to one and the same distortion gage field. Keeping in mind the aforementioned, our interest in the present article is to complete this formalism; in particular, much more can be done to clarify and rigorously formulate the results and formulations already provided if we uniquely determine the 'distortion' transformations and, thus, the 'distortion' group. This can be achieved by a non-linear realization of the Lie group G of 'distortion' of the G_{12} in the framework of the method of phenomenological Lagrangians Coleman & Zumino (1969)-Isham (1969). Conceptually and technically, this method is versatile and powerful. This allows, for example, to treat chiral dynamics as the theory of spontaneous breaking of chiral symmetry and/or General Relativity as the theory of spontaneous breaking of affine and conformal symmetries Borisov (1974). In what follows, we study the geometrical structure of the space of parameters in terms of Cartan forms , implying the appropriate Maurer-Cartan's structure equations. In its present formulation, we identify the 'distortion' gage fields with the Goldstone fields and calculate the conserved currents. However, the formalism being followed may seem a purely formal exercise. To avoid this unnecessary misimpression, as a tangible example indicating the potential insights this formalism may yield, we refer to Ter-Kazarian (1997, 2001) for further details. The paper consists of four parts. Section 2 deals with the space G_{12} and its distortion, section 3 considers the spontaneous breaking of 'distortion' symmetry. Section 4 deals with the Lagrangian of Goldstone fields and conserved currents. We will be brief and often suppress the indices without notice.

*gago_50@yahoo.com

doi: https://doi.org/10.52526/25792776-26.73.1-17

2. The space G_{12}

2.1. General prescription

Consider the curve $\lambda(t) : R^1 \to G_{12}$ passing through the point $p = \lambda(0) \in G_{12}$ with tangent vector $\mathbf{A}|_{\lambda(t)}$. The $\{\zeta\}$ are local coordinates in an open neighborhood of $p \in \mathcal{U}$. The 12-dimensional smooth vector field $\mathbf{A}_p = \mathbf{A}(\zeta)$ belongs to the section of tangent bundle $\mathbf{T}_p$ at the point $p(\zeta)$. The one parameter group of diffeomorphisms A^t for the curve $\zeta(t)$ passing through the points p and $\zeta(0) = \zeta_p$, $\quad \dot{\zeta}(0) = \mathbf{A}_p$, is written as

$$d\,A^t_p(\mathbf{A}) = \frac{d}{d\,t}\bigg|_{t=0} A^t\,(\zeta(t)) = \mathbf{A}_p(\zeta),$$

such that

$$d\,A^t : \mathbf{T}(G_{12}) \to R\left(\mathbf{T}(G_{12}) = \bigcup_{p(\zeta)} \mathbf{T}_p\right).$$

The $e_{(\lambda,\mu,\alpha)} = O_{\lambda,\mu} \times \sigma_\alpha \qquad (\lambda, \mu = 1, 2; \quad \alpha = 1, 2, 3)$ are linear independent 12 unit basis vectors at the point $p \in G_{12}$ provided by the unit vectors $O_{\lambda,\mu}$ and σ_α implying

$$< O_{\lambda,\mu}, O_{\tau,\nu} >= {}^*\delta_{\lambda\tau}{}^*\delta_{\mu\nu}, \quad < \sigma_\alpha, \sigma_\beta >= \delta_{\alpha\beta}, \tag{1}$$

where $\delta_{\alpha\beta}$ is the Kronecker symbol, but ${}^*\delta_{11} = {}^*\delta_{22} = 0$ and ${}^*\delta_{12} = {}^*\delta_{21} = 1$. The basis vectors σ_α refer to given three dimensional ordinary space $R^3_{\lambda\mu}$. The metric on G_{12} reads

$$\hat{\mathbf{g}} : \mathbf{T}_p \times \mathbf{T}_p \to C^\infty(G_{12}).$$

Let also the massless gauge field $a_A(\zeta)$ associate with the connection in the principal bundle $p : E \to G_{12}$ with a structure group U^{loc}, where the coordinates ζ exist in the whole region $\mathcal{U}$ of space G_{12}.

2.2. Distortion of the G_{12}

Determining the distortion transformations of basis vectors e, given at point $p \in G_{12}$, by matrix $D(a)$ of 'distortion' group G, we assume that the basis vectors O undergo distortion transformations generated by the matrix $\mathcal{Q}(a)$ of the group Q

$$O_A(a) = \mathcal{Q}^{\tau,\nu}_A(a)\,O_{\tau,\nu}, \quad \mathcal{Q}^{\tau,\nu}_A(a) = \delta^\tau_\lambda\delta^\nu_\mu + æ\,a_A{}^*\delta^\tau_\lambda{}^*\delta^\nu_\mu, \tag{2}$$

where summation over repeated indices is implied, æ is the gauge coupling constant. To avoid from the proliferation of indices, hereafter the following convention (A) will be observed in indexing $(\lambda\mu\alpha)$, etc. Note that the distortion transformations Eq. (2) violate the first condition in Eq. (1). The basis vectors σ, in turn, undergo distortion transformations of rotation: $\sigma_A(\theta) = \mathcal{R}^\beta_A(\theta)\,\sigma_\beta$, where $\quad \mathcal{R}(\theta) \in SO(3)$ is the element of group of rotations of planes involving two arbitrary axes around orthogonal third axes in the given three-dimensional ingredient space $R^3_{\lambda\mu} \quad (G_{12} = \sum_{\lambda\mu} \oplus R^3_{\lambda\mu})$. Even though, the rotation $\mathcal{R}(\theta)$ violates the second relation of orthogonality in Eqs. (1) between the basis vectors σ of different ingredient spaces, however, the three-dimensionality of each ingredient space remains intact. The resulting rotation matrix $\mathcal{R}(\theta)$ should be independent of the sequence of rotations carried out. This implies a symmetrization of resulting matrix $\mathcal{R}(\theta) = \frac{1}{6}\sum_{i\neq j\neq k} \mathcal{R}^{(ijk)}(\theta)$, where $\mathcal{R}^{(ijk)}(\theta) \in SO(3)$ is the matrix of rotations carried out in the given sequence $(ijk) \quad (i, j, k = 1, 2, 3)$. The infinitesimal transformations read

$$\delta\,\mathcal{Q}^{\tau,\nu}_A(a) = æ\,\delta\,a_A X^{\tau\nu}_{\lambda\mu} \in Q, \quad \delta\,\mathcal{R}(\theta) = -\tfrac{i}{2} M_{\alpha\beta}\delta\,\omega^{\alpha\beta} \in SO(3), \tag{3}$$

provided by the generators $X^{\tau\nu}_{\lambda\mu} = {}^*\delta^\tau_\lambda{}^*\delta^\nu_\mu$, and $I_i = \frac{\sigma_i}{2}$, where σ_i are the Pauli's matrices, that

$$M_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}I_\gamma, \quad \left[\tfrac{\sigma_i}{2}, \tfrac{\sigma_k}{2},\right] = i\varepsilon_{ikj}\tfrac{\sigma_j}{2}, \quad \delta\,\omega^{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}\delta\,\theta_\gamma. \tag{4}$$

One may now implement the matrix

$$D(a,\,\theta) = \mathcal{Q}(a) \times \mathcal{R}(\theta) \tag{5}$$

doi: https://doi.org/10.52526/25792776-26.73.1-17

of 'distortion' group $G = Q \times SO(3)$ by which the basis vector e is transformed at point $p \in G_{12}$. The infinitesimal transformations of the group G read

$$\begin{array}{l} D_{(d\,a^A,\,d\,\theta^A)} = I + d\,D_{(a^A,\,\theta^A)}, \\ d\,D_{(a^A,\,\theta^A)} = i\left[d\,a^A X_A + d\,\theta^A I_A\right], \end{array} \tag{6}$$

where $I_A \equiv I_\alpha$ at given λ and μ $(A = (\lambda\mu\alpha))$. We associate with each point $(a,\,\theta)$ of the group space the Cartesian basis vector, in group sense, that corresponds to the basis vector at the point of origin $(0,\,0)$

$$\begin{array}{l} G(g) \equiv G(a + d\,a,\,\theta + d\,\theta)\,G^{-1}(a,\,\theta) = \\ G(d\,a',\,d\,\theta')\,G^{-1}(0,\,0) = G(d\,a',\,d\,\theta'). \end{array} \tag{7}$$

It is introduced to ensure that the vector $(a,\,\theta;\,a + d\,a,\,\theta + d\,\theta)$ has the same analytical expression of the vector $(0,\,0;\,d\,a',\,d\,\theta)'$ (Eq. (6))

$$D_{(a,\,\theta)}\,d\,D^{-1}_{(a,\,\theta)} = i[\omega^A\,X_A + \vartheta^A\,I_A], \tag{8}$$

where we denote $d\,a'^A = \omega^A(a,\,\theta;\,d\,a,\,d\,\theta)$ and $d\,\theta'^A = \vartheta^A(a,\,\theta;\,d\,a,\,d\,\theta)$. In accordance, an infinitesimal action of the group on basis vectors e_A can be written as

$$\begin{array}{l} d\,e_A = e_A\,d\,F_A = O_A(d) \times \sigma_A + O_A \times \sigma_A(d) = \\ i[\omega^A(d)\,X_A + \vartheta^A(d)\,I_A]\,e_A, \end{array} \tag{9}$$

where we denote $e_A \equiv (\exp F_A)$. The functions $\omega^A(d)$ and $\vartheta^A(d)$ will be determined in the next section. One may, however, envisage that the distortion transformations of basis vectors O and σ, in fact, are not independent but rather governed by a spontaneous breaking of 'distortion' symmetry, which yields the uniquely determined matrix Eq. (5).

3. A spontaneous breaking of 'distortion' symmetry

Our major goal is now to proceed to the spontaneous breaking of 'distortion' symmetry. Although the generators X_A (Eq. (3)) of the group Q cannot be considered the generators of the quotient space G/H, because they do not complete the group H to the dynamical group G, and therefore the 'distortion' fields $a_A(\zeta)$ cannot be identified directly with the Goldstone fields of the spontaneous breaking of the 'distortion' symmetry. However, the modified 'distortion' fields can be identified with the Goldstone fields. Actually, let us deal with the non-linear realizations of the 'distortion' group G. Following a general prescription of the method of phenomenological Lagrangians Coleman & Zumino (1969), Curtis G. & Zumino (1969), Isham (1969), Weinberg (1970), in the case at hand we treat the 'distortion' group G and its stationary subgroup $H = SO(3)$, respectively, as the dynamical group and its algebraic subgroup, respectively, and then let the parameters $\overline{a}_A(a)$ of the quotient space G/H of adjacent classes be the Goldstone fields of the spontaneous breaking of the 'distortion' symmetry. In order to obtain a feeling for this point, we re-define, for example, the pair of basis vectors O_{11} and O_{22} in terms of new orthogonal unit vectors n_1 and n_2

$$O_{11} = \tfrac{1}{\sqrt{2}}\,(n_1 + n_2)\,, \quad O_{22} = \tfrac{1}{\sqrt{2}}\,(n_1 - n_2)\,, \tag{10}$$

where $n_1^2 = -n_2^2 = 1, \quad < n_1,\,n_2 >= 0$ (see Eq. (1)). Denoting $\tan\,\overline{\theta}_A = -æ\,a_A$, it is straightforward to show that the distortion transformations Eq. (2) of basis vectors O_{11} and O_{22} are reduced to the transformations of the group $SO(3)$ of modified basis vectors

$$\overline{O}_A = \frac{1}{\sqrt{2}}\left(n_1(\overline{\theta}_A) + \epsilon_A\,n_2(\overline{\theta}_A)\right) = \cos\,\overline{\theta}_A O_A, \tag{11}$$

where $\epsilon_{(11\alpha)} = -\epsilon_{(22\alpha)} = 1$, such that

$$\begin{pmatrix} n_1(\overline{\theta}_A) \\ n_2(\overline{\theta}_A) \end{pmatrix} \begin{pmatrix} \cos\,\overline{\theta}_A & \sin\,\overline{\theta}_A \\ -\sin\,\overline{\theta}_A & \cos\,\overline{\theta}_A \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}. \tag{12}$$

Here the rotation on $\overline{\theta}_{(11\alpha)}$ is in clockwise direction, while it is counterclockwise for $\overline{\theta}_{(22\alpha)}$. The same holds for the pair O_{12} and O_{21}. The 'distortion' group $G = Q \times SO(3)$, therefore, can be mapped onto the

doi: https://doi.org/10.52526/25792776-26.73.1-17

group $G = SO(3) \times SO(3)$ in a one-to-one manner. The latter is isomorphic to the isotopic chiral group $SU(2) \times SU(2)$, for which the method of phenomenological Lagrangians is well known Coleman & Zumino (1969), Curtis G. & Zumino (1969), Isham (1969), Weinberg (1970). For the sake of simplicity, throughout this subsection we leave the Greek indices implicit unless otherwise stated, such that $A = (\lambda\mu i) \rightarrow i = 1, 2, 3,$. But it goes without saying that all the results obtained refer to the given three-dimensional ingredient space $R^3_{\lambda\mu}$. Three I_i among the six generators of the group correspond to isotopic transformations, and three K_i correspond to special chiral transformations mixing the states of different parities. They imply the conventional commutation relations

$$[I_i,\, I_j] = i\varepsilon_{ijk} I_k; \quad [I_i,\, K_j] = i\varepsilon_{ijl} K_l; \quad [K_i,\, K_j] = i\varepsilon_{ijk} I_k, \tag{13}$$

of the invariant subgroup $H = SO(3)$, with the generators I_i, and of the quotient space G/H, with the generators K_i, where ε_{ijk} denotes the antisymmetric unit tensor. Three parameters $\overline{a}^i(a)$ of special chiral transformations, with respect to which the Lagrangian of physical fields is not invariant, can be identified as three Goldstone fields. They are introduced to make provisions for Eq. (9), which, when incorporated into Eq. (11), are designed to yield

$$\begin{aligned} &d\,\overline{e}_A(\overline{a}) = \overline{e}_A(a)\, d\,\overline{F}_A(\overline{\theta},\, \theta) = \overline{O}_A(d) \times \sigma_A + \overline{O}_A \times \sigma_A(d) = \\ &i[\omega^i(\overline{a},\, d\,\overline{a})\, K_i + \vartheta^l(\overline{a},\, d\,\overline{a})\, I_l]\, \overline{e}_A(\overline{a}) \end{aligned} \tag{14}$$

in terms of generators of the group $SU(2) \times SU(2)$, where $\overline{e}_A(\overline{a}) = \overline{O}_A(\overline{a}) \times \sigma_A(\overline{a})$. Now we are looking for the function $\overline{\theta}(\theta)$ for which $d\,\overline{F}_A(\overline{\theta},\, \theta)$ constitutes a total differential and, thus, $\omega^i(\overline{a},\, d\,\overline{a})$ and $\vartheta^l(\overline{a},\, d\,\overline{a})$ are the Cartan's forms. Then

$$\frac{\partial^2 \overline{F}_A}{\partial\,\overline{\theta}\,\partial\,\theta} = \frac{\partial^2 \overline{F}_A}{\partial\,\theta\,\partial\,\overline{\theta}}, \tag{15}$$

where we suppress the indices without notice ($\overline{\theta} \equiv \overline{\theta}_A$ and $\theta \equiv \theta_A$). Using the explicit form of infinitesimal transformations (Eq. (3))

$$d\,\sigma_{(\lambda\mu l)} = \sigma_{(\lambda\mu l)}(d) = \tfrac{1}{2}\varepsilon_{lkj}\sigma_k d\,\theta_{(\lambda\mu j)}, \tag{16}$$

it is straightforward to calculate the partial derivative

$$\frac{\partial\, F_A}{\partial\,\theta} = \frac{\partial\,\sigma_A}{\sigma_A\,\partial\,\theta} \equiv \frac{\sigma_A(\partial)}{\sigma_A\,\partial\,\theta} = \frac{1}{\sin\,\theta_A}. \tag{17}$$

The similar relation holds for the vectors $d\,n_{1A}(\overline{\theta})$ and $d\,n_{2A}(\overline{\theta})$ (Eq. (12)), and thus $\frac{\partial\, F_A}{\partial\,\overline{\theta}} = \frac{1}{\sin\,\overline{\theta}_A}$. Hence , the holonomy condition Eq. (15) is reduced to

$$\frac{\partial}{\partial\,\theta}\left(\frac{1}{\sin\,\overline{\theta}_A}\right) = \frac{\partial}{\partial\,\overline{\theta}}\left(\frac{1}{\sin\,\theta_A}\right), \tag{18}$$

the nontrivial solution of which is $\theta_A = \overline{\theta}_A$, and thus, we arrive at

$$\tan\,\theta_A = -æ\, a_A. \tag{19}$$

This key relation contains no new information, beyond the fact that it uniquely determines the 12 angles θ_A of rotations around each of the 12 axes (A) at a given distortion field a_A. We are now in a position to infer the Maurer-Cartan's structure equations. According to Poincaré's theorem, the external derivative $(')$ of the total differential form is zero, and therefore Eq. (14) gives

$$\begin{aligned} &(\overline{e}_A(a)\, d\,\overline{F}_A)' = \overline{e}_A(a)\, \left([d\,\overline{F}_A, \delta\,\overline{F}_A] + (d\,\overline{F}_A)'\right) = \\ &i\left([\omega^i(\overline{a},\, d\,\overline{a})\, K_i + \vartheta^l(\overline{a},\, d\,\overline{a})\, I_l]\, \overline{e}_A(a)\right)' = 0, \end{aligned} \tag{20}$$

where $[d\,\overline{F}_A, \delta\,\overline{F}_A] = (d\,\overline{F}_A)' = 0$. In the calculation of the external differential and external product of the forms in Eq. (20), the differentials of functions $(K_i\,\overline{e}_A)$ and $(I_l\,\overline{e}_A)$ figured in the bilinear differential

$$\begin{aligned} &\delta\,d\,\overline{e}_A = i[\delta\,\omega^i(d)\, X_i + \delta\,\vartheta^l(d)\, I_l]\,\overline{e}_A + \\ &i[\omega^i(d)\,\delta(K_i\,\overline{e}_A) + \vartheta^l(d)\,\delta(I_l\,\overline{e}_A)], \end{aligned} \tag{21}$$

will be defined according to Eq. (14). Equating the coefficients of the same linearly independent generators of the external differential in Eq. (20) to zero yields the following system of structure equations:

$$\begin{aligned} &(\omega^i)' = [\omega^k,\, \vartheta^\beta]\,\varepsilon_{ik\beta}, \\ &(\vartheta^\gamma)' = [\vartheta^\alpha,\, \vartheta^\beta]\,\varepsilon_{\gamma\alpha\beta}/2 + \varepsilon_{\gamma ki}[\omega^k,\, \omega^i]/2, \end{aligned} \tag{22}$$

doi: https://doi.org/10.52526/25792776-26.73.1-17

provided we have the external differential and external product of the forms written as

$$
\begin{aligned}
&(\omega^i)' = -\delta\,\omega^i(d) + d\,\omega^i(\delta),\\
&\left[\omega^k,\,\vartheta^\beta\right] = \omega^k(d)\vartheta^\beta(\delta) - \omega^k(\delta)\vartheta^\beta(d).
\end{aligned}
\tag{23}
$$

Defining the new forms $\omega^i_k = \vartheta^\beta\varepsilon_{ik\beta}$ and using the relations $\varepsilon_{k\alpha\beta}\,\varepsilon_{ijk} = (\varepsilon_{i\beta k}\,\varepsilon_{k\alpha j} - \varepsilon_{i\alpha k}\,\varepsilon_{k\beta j})$, which stem from Yacobi's identity.

$$
\{[K_i[I_\alpha,\,I_\beta]]\} + \quad \text{permutations} \equiv 0,
\tag{24}
$$

the Eqs. (22) yield the Maurer-Cartan's structure equations

$$
\begin{aligned}
&(\omega^i)' = [\omega^k,\,\omega^i_k],\\
&(\omega^i_k)' = -R^l_{jki}[\omega^k,\,\omega^i]/2 + [\omega^k_j,\,\omega^l_k],
\end{aligned}
\tag{25}
$$

where $R^l_{jki} = -\varepsilon_{lj\gamma}\varepsilon_{\gamma ki}$ is the curvature of the group space, These equations describe the motion of the orthogonal reper joint to the given point of the group space. The forms ω^i and ω^i_k are interpreted as the transformations of translation and rotation of the orthogonal reper, such that the rotation $(\vartheta^l I_l)$ belongs to the stationary group H, while the translation reads as $(\omega^i K_i)$. Whereas the invariant constraint Eq. (18) means that there always exists a rotation transformation of the stationary subgroup H annulling the change in the equation of Cartan's forms arising from the translation transformation of the quotien space $\overline{Q} = G/H$. A general transformation of the group G can be written as

$$
G = \overline{Q}(\overline{a})H(\theta),
\tag{26}
$$

where $\overline{Q}(\overline{a})$ is the transformation belonging to the left adjacent class G/H of the group G by subgroup H. Acting from the left by the group element g we define the transformation of parameters $\overline{a}$ and θ:

$$
G(g)\overline{Q}(\overline{a})H(\theta) = \overline{Q}(\overline{a}'(\overline{a},\,g))H(\theta'(\theta,\,\overline{a},\,g)).
\tag{27}
$$

The Cartan's forms allow an exponentiation of the group element $\overline{Q}(a) = \exp(i\overline{a}^i K_i)$, which determines the modified 'distortion' fields $\overline{a}(a)$

$$
d\,\overline{e}_A(\overline{a}) = \left[\exp(-i\overline{a}^i K_i)\,d\,\exp(i\overline{a}^i K_i)\right]\,\overline{e}_A(\overline{a}).
\tag{28}
$$

Exponential parametrization of the finite transformations of the group is equivalent to the choice of normal coordinates in the quotien space G/H. The solution of the structure equations (25) reads

$$
\begin{aligned}
&\omega^i(\overline{a},\,d\,\overline{a}) = (\sin\sqrt{m}/\sqrt{m})^i_k\;d\,\overline{a}^k, \quad m^i_k = -\varepsilon_{ij\alpha}\varepsilon_{\alpha kl}\,\overline{a}^j\,\overline{a}^k,\\
&\vartheta^\alpha(\overline{a},\,d\,\overline{a}) = [(1-\cos\sqrt{m})\,/m]^i_k\;d\,\overline{a}^k\,\varepsilon_{\alpha il}\,\overline{a}^l,.
\end{aligned}
\tag{29}
$$

Using the relations

$$
\left(m^{(n)}\right)^i_j = (\overline{a}^2)^{n-1}\left(m^{(1)}\right)^i_j, \quad m^i_j = \varepsilon_{ilk}\varepsilon_{kjn}\overline{a}^l\overline{a}^n = \overline{a}^2\delta_{ij} - \overline{a}^i\overline{a}^j,
\tag{30}
$$

the Eq. (29) was reduced as

$$
\begin{aligned}
&\omega^i_A = \omega^i(\overline{a},\,\partial_A\,\overline{a}) = \partial_A\,\overline{a}^i + (\delta_{ik} - \overline{a}^i\overline{a}^k/\overline{a}^2)\left(\sin\sqrt{\overline{a}^2}/\sqrt{\overline{a}^2} - 1\right)\partial_A\,\overline{a}^k,\\
&\vartheta^i_A = \vartheta^i(\overline{a},\,\partial_A\,\overline{a}) = \left[\left(1-\cos\sqrt{\overline{a}^2}\right)/\overline{a}^2\right]\partial_A\,\overline{a}^l\,\varepsilon_{ilj}\,\overline{a}^j.
\end{aligned}
\tag{31}
$$

4. Conserved currents

The Lagrangian of Goldstone field $(\overline{a})$ can be identified with the square of the interval of the geodesic line, with a minimal number of derivatives, between the infinitely close points $\overline{a}^i$ and $\overline{a}^i + d\,\overline{a}^i$

$$
L_{\overline{a}}(\zeta) = \tfrac{1}{2}\,\omega^i(\overline{a},\,\partial_A\,\overline{a})\omega^i(\overline{a},\,\partial_A\,\overline{a}),
\tag{32}
$$

where $\varepsilon_{\alpha il}\varepsilon_{l\alpha j} = -\delta_{ij}$ is the metric tensor of the group space. In normal coordinates, it takes the form

$$
\begin{aligned}
&L_{\overline{a}}(\zeta) = (\partial_A\,\overline{a})^2/2 +\\
&(\delta_{ik} - \overline{a}^i\overline{a}^k/\overline{a}^2)\left(\sin^2\sqrt{\overline{a}^2}/\sqrt{\overline{a}^2} - 1\right)\partial_A\,\overline{a}^i\,\partial_A\,\overline{a}^k/2.
\end{aligned}
\tag{33}
$$

doi: https://doi.org/10.52526/25792776-26.73.1-17

Since the massless gauge field (a) is associated with the gauge group U^{loc}, the Lagrangian Eq. (34) should be equated to undegenerated Killing form defined on the Lie algebra of the group U^{loc} in adjoint representation

$$L_{\overline{a}}(\zeta) = L_a(\zeta) = -\tfrac{1}{4} < F_{AB}(a),\, F^{AB}(a) >_K, \tag{34}$$

where $F_{AB}(a)$ is the antisymmetric tensor of the gauge field (a). The Goldstone field $(\overline{a})$ as a function of the gauge field (a) can be determined from Eq. (34). Suppose Ψ is a collection of matter fields of arbitrary spin , which are sections of the vector bundles associated with U^{loc} by mapping $\Psi : G_{12} \to E$ that $p\,\Psi(\zeta) = \zeta$. The various suffixes of Ψ are left implicit. They take values in the standard fiber $\mathcal{F}_\zeta$ upon $\zeta : p^{-1}(\mathcal{U}) = \mathcal{U} \times \mathcal{F}_\zeta$, where an expansion into the direct product $p^{-1}(\mathcal{U}) = \mathcal{U} \times G_{12}$ is defined upon $\mathcal{U}$. The fiber is the Hilbert vector space on which the linear representation $U(\zeta)$ of the group U^{loc} is given. This space can be regarded as the Lie algebra of the group U^{loc} upon which the Lie algebra acts according to the law of the adjoint representation: $a \leftrightarrow ad\,a\;\; \Psi \to [a\,,\Psi]$. The covariant derivatives of the matter fields Ψ interacting with the Goldstone fields $\overline{a}$ are determined by means of the form ϑ^α

$$L = L_0\,(\Psi,\, \partial_A\Psi + \vartheta^\alpha(\overline{a},\, \partial_A\,\overline{a})T_\alpha\,\Psi)\,, \tag{35}$$

where $L_0\,(\Psi,\, \partial_A\Psi)$ is the Lagrangian of interacting matter fields classified by the linear representations T_α of the subgroup H. The Lagrangians Eq. (33) and Eq. (35) are invariant with respect to 'distortion' translations and rotations. These also should be completed by the transformations

$$\Psi' = (\exp\,[i\eta'^\alpha(\overline{a},\, g)T_\alpha])\;\Psi, \tag{36}$$

where the parameters $\eta'^\alpha(\overline{a},\, g)$ can be determined from Eq. (27). In exponential parametrization this gives

$$G(g)\exp(i\overline{a}^i K_i)\exp(i\eta^\alpha I_\alpha) = \exp\left[i\overline{a}^i(\overline{a},\, g)K_i\right]\exp\left[i\eta'^\alpha(\overline{a},\, g)I_\alpha\right]. \tag{37}$$

According to the Eq. (37), the transformation of parameters $\overline{a}$ and θ at the infinitesimal translation $G(g) = (1 + i\varepsilon^i\,K_i) + O(\varepsilon^2)$ can be written

$$(1 + i\varepsilon^i\,K_i)\exp(iK_i\overline{a}^i) = \exp\left[iK_i(\overline{a}^i + \delta\,\overline{a}^i(\overline{a},\,\varepsilon))\right]\exp\left[i\delta\,\theta^\alpha(\overline{a},\,\varepsilon))I_\alpha\right]. \tag{38}$$

Employing the Feynman's method of ordering by means of auxiliary parameter t Gell-Mann (1960) which implies $\int_x^y\, A(t)d\,t = A(y-x)$, in standard manner we expand

$$\begin{gathered}\exp\left[iK_i(\overline{a}^i + \delta\,\overline{a}^i)\right] = \exp(iK_i\overline{a}^i) + i\exp(iK_i\overline{a}^i)\times \\ \times\int_0^1\, d\,t\,\exp(-iK_i\overline{a}^i\,t)\,(\delta\,\overline{a}^i\,K_i)\,\exp(iK_i\overline{a}^i\,t) + \cdots\,.\end{gathered} \tag{39}$$

Then the Eq. (38) gives

$$i\varepsilon^i\,K_i(1) = i\int_0^1\, d\,t\delta\,\overline{a}^i K_i(t) + i\delta\,\theta^\alpha(\overline{a},\,\theta)I_\alpha, \tag{40}$$

where

$$K_j(t) = \exp(-iK\overline{a}^i t)K_j\exp(iK_i\overline{a}^i t). \tag{41}$$

In analogy defining the $I_\alpha(t)$, and taking derivatives of the $K_j(t)$ and $I_\alpha(t)$ with respect to parameter t, we get

$$\begin{aligned}\tfrac{\partial}{\partial t}\,K_j(t) &= i\varepsilon_{\alpha j i}\overline{a}^i\,I_\alpha(t); \quad K_j(0) = K_j, \\ \tfrac{\partial}{\partial t}\,I_\alpha(t) &= i\varepsilon_{l\alpha k}\overline{a}^k\,K_l(t); \quad I_\alpha(0) = 0,\end{aligned} \tag{42}$$

the solution of which can be written in the form of Eq. (29)

$$K_j(t) = (\cos\,\sqrt{m}\,t)^i_j\;K_j + i\,(\sin\,\sqrt{m}\,t/\sqrt{m})^l_j\;\varepsilon_{\alpha l k}\,\overline{a}^k\,I_\alpha. \tag{43}$$

Inserting this solution into the Eq. (40) and equating the coefficients at the same generators $K_j I_\alpha$, one gets

$$\begin{aligned}&\delta\,\overline{a}^i(\overline{a},\,\theta)) = -(\sqrt{m}\;\coth\,\sqrt{m})^i_k\varepsilon^k + O(\varepsilon^2), \\ &\delta\,\theta^\alpha(\overline{a},\,\theta)) = \{[\sin\,\sqrt{m} - \coth\,\sqrt{m}(1-\cos\,\sqrt{m})]/\sqrt{m}\}^l_i\,\varepsilon^i\,\varepsilon_{\alpha l k}\overline{a}^k.\end{aligned} \tag{44}$$

The transformation of parameters in the case of rotation $G(g') = (1 + i\xi^\alpha\,I_\alpha) + O(\xi^2)$ is obtained in the same way

$$\delta\,\overline{a}^i(\xi)) = -\varepsilon_{i\alpha k}\,\xi^\alpha\,\overline{a}^k + O(\xi^2), \quad \delta\,\theta^\alpha(\overline{a},\,\xi)) = \xi^\alpha. \tag{45}$$

doi: https://doi.org/10.52526/25792776-26.73.1-17

Let the fields undergo the infinitesimal transformations

$$\Phi^j(\zeta) \to \Phi^j(\zeta) + \varepsilon(\zeta)^k \prod_k^j(\Phi), \tag{46}$$

where $\prod_k^j(\Phi)$ is the non-linear function of the fields. The current related to transformation Eq. (46)

$$J_k^A = -\delta L(\Phi, \partial \Phi)/\delta(\partial_A \varepsilon^k) = -\prod_k^j(\Phi)\, \delta L(\Phi, \partial \Phi)/\delta(\partial_A \Phi^j), \tag{47}$$

can be written

$$\partial_A J_k^A = -\delta L(\Phi, \partial \Phi)/\delta \varepsilon^k. \tag{48}$$

Thus, if the Lagrangian is invariant with respect to constant transformations Eq. (46), then the corresponding current is conserved. Leaving the 12-dimensional indices A implicit, the conserved currents corresponding to Eq. (29), Eq. (44) and Eq. (45) can be calculated in normal coordinates as

$$\begin{aligned} J^k(\overline{a}, \partial \overline{a}) &\equiv \omega^k(2\overline{a}, \partial \overline{a}) = (\sin 2\sqrt{m}/2\sqrt{m})_i^k \partial \overline{a}^i, \\ J^\alpha(\overline{a}, \partial \overline{a}) &\equiv \vartheta^\alpha(2\overline{a}, \partial \overline{a}) = -\left[(1 - \cos 2\sqrt{m})\,/2m\right]_k^i \, \partial \overline{a}^k \, \varepsilon_{\alpha i l}\, \overline{a}^l. \end{aligned} \tag{49}$$

5. Concluding remarks

It is worth reflecting briefly on the results obtained so far. Based on the powerful method of phenomenological Lagrangians, the new conceptual element in non-linear realizations of the Lie group G of 'distortion' of the space G_{12}, is noteworthy. This restores the most important unifying feature of the theory and completes the GGP formalism; in particular, it clarifies and rigorously establishes the results and formulations already given in Ter-Kazarian (1997, 2001). We have identified the parameters of quotien space G/H of adjacent classes by the subgroup H with the Goldstone fields of spontaneous breaking of 'distortion' symmetry. The inferred key relation Eq. (19) uniquely determines the 12 angles θ_A of rotations around each of the 12 axes at a given distortion field a_A, due to which the distortion transformations and, thus, the distortion group G become uniquely determined. We have studied the geometrical structure of the space of parameters in terms of Cartan forms, implying the appropriate Maurer-Cartan's structure equations, which allow us to infer the group invariants and calculate the conserved currents.

References

Borisov A.B. O. V., 1974, Theor. and Math. Phys., 21, 329

Coleman S. W. J., Zumino B., 1969, Phys.Rev., 177, 2239

Curtis G. Callan J. C. S. W. J., Zumino B., 1969, Phys.Rev., 177, 2247

Gell-Mann M. Levi M., 1960, Nuovo Cimento, 16, 705

Isham C., 1969, Nuovo Cimento, Sec. A 59, 356

Ter-Kazarian G., 1997, Nuovo Cimento, Sec. B 112

Ter-Kazarian G., 2001, J. Phys. Soc. Japan, Suppl. B 70, 84

Weinberg S., 1970, Lectures on elementary particles and quantum field theory. Cambridge

doi: https://doi.org/10.52526/25792776-26.73.1-17